Math 211

Final Exam

Name __________________________
I. Logic and proof

1. (15 pts.) Consider the statement "If all vertices of a connected graph have even degree then the graph has an Euler circuit"

What is the sufficient condition?

What is the necessary condition?

What is the converse of the statement?

What is the contrapositive of the statement?

As it happens, the converse of this statement is also true. Re-write the statement using the words "sufficient" and "necessary" to indicate this.
2. (10 pts). Construct a truth table for \(((\neg p) \land (p \Rightarrow q)) \Rightarrow p\). Is the statement a tautology?

3. (10 pts.)

a. If \(p(x)\) means "\(x\) is a point", \(l(x)\) "\(x\) is a line", \(\text{par}(x,y)\) "\(x\) and \(y\) are parallel" and \(\text{on}(x,y)\) "\(x\) is on \(y\)" , translate into ordinary English (i.e., not simply a transliteration):
   \[\forall x \forall y (p(x) \land l(y) \land \neg \text{on}(x,y) \Rightarrow \exists z (l(z) \land \text{on}(x,z) \land \text{par}(y,z)))\]

b. Suppose \(F(x)\) means "\(x\) is a bird", \(B(x)\) "\(x\) is brightly colored", and \(G(x)\) "\(x\) is ground-feeding" Translate the statement "No brightly colored bird is a ground feeder" into logical notation.
4. (5 pts.) Simplify

\[ \neg \forall x (S(x) \Rightarrow T(x)) \]

so that not (\(\neg\)) does not appear

5. (10 pts.) Prove that \(A \lnot (B \cup C) = (\lnot A \lnot B) \cup (A \lnot C)\)
II. Algorithms, functions, and asymptotic behavior

1. (5 pts.) Give a brief definition of an algorithm.

2. (10 pts.)

What do we mean by a function f: A -> B?

Let f(x) = x + 1 and let g(x) = 2x. What is

(f \circ g)(x)\?

(f + g)(x)\?
3. (10 pts.) Give both a formal and an informal definition of what it means to say that $f$ is $O(g)$

4. (10 pts.) For each of the following algorithms, give the best "big-O" estimate of its cost in time as a function of the size of the array or graph:

   linear search

   bubble sort

   binary search

   Finding a shortest path between two vertices of a graph

   Finding a Hamilton circuit in a graph

   QuickSort
III. Number theory (with some incorporated material on relations)

1. (10 pts.) Consider the integers mod 5. We define

\[ [k] = \{ n \in \mathbb{Z} \mid n \equiv k \mod 3 \} \]

a. Why are there only five distinct such sets ([0], [1], [2], [3], [4])

b. We can define addition and multiplication on these three sets as follows:

\[ [j] + [k] = [j+k] \]
\[ [j] * [k] = [j*k] \] (as in a lecture past)

Complete the following addition and multiplication tables using only [0], [1], [2], [3], and [4]:

<table>
<thead>
<tr>
<th>+</th>
<th>[0]</th>
<th>[1]</th>
<th>[2]</th>
<th>[3]</th>
<th>[4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[1]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[2]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[3]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[4]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>*</th>
<th>[0]</th>
<th>[1]</th>
<th>[2]</th>
<th>[3]</th>
<th>[4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[1]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[2]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[3]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[4]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is 1/[2]?
2. (10 pts.)
   a. Define \(a | b\) (evenly divides)

   b. Define a relation on the integers by \(a | b\). Describe the properties of this relation (i.e., symmetric, anti-symmetric, transitive, etc.)

IV. Induction

1. (10 pts.) Prove that \(\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}\) where \(f_k\) is the \(k^{th}\) Fibonacci number \((f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, \ldots)\). Carefully describe each step.
V. Counting and probability

1. (5 pts.) What is the coefficient of $x^7y^3$ in the expansion of $(x + y)^{10}$ (please provide a numeric answer).

2. (10 pts.) In how many ways can you throw a 4 using a pair of fair dice? What is the probability of doing so?
3. (10 pts.)

How many ways can you draw a five-card hand from a deck of 52 cards so that the hand contains exactly one ace?

What is the probability of doing so?

4. (10 pts.) In how many ways can you place 10 indistinguishable marbles into three boxes?

At least one box has at least how many marbles? What principle are you using?
VI Relations and databases (10 pts.)

a. (10 pts.) What does it mean for a relation $R \subseteq AXB$ to be

reflexive

symmetric

anti-symmetric

transitive

an equivalence relation
b. (15 pts.) Consider the usual supplier-parts-project database

\[
\begin{align*}
S(SNO, SNAME, STATUS, CITY) \\
P(PNO, PNAME, COLOR, WEIGHT, CITY) \\
J(JNO, JNAME, CITY) \\
SPJ(SNO, PNO, JNO, QTY)
\end{align*}
\]

In the relational algebra,

What does the **projection** operator do?

What does the **selection** operator do?

What does the **join** operator do?

c. In the SQL statement below, identify occurrences of the **projection**, **selection**, and **join** operations:

\[
\begin{align*}
\text{SELECT SNAME} \\
\text{FROM S, P, SPJ} \\
\text{WHERE P.COLOR = "RED"} \\
\text{AND P.PNO = SPJ.PNO} \\
\text{AND S.SNO = SPJ.SNO;}
\end{align*}
\]
VII. Graphs

1. (10 pts.) Sketch the graph resulting from the following adjacency matrix:

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
\end{bmatrix}
\]

The adjacency matrix counts the number of paths of length 1 between vertices. Calculate the matrix that counts the number of paths of length 2 between vertices.

Construct an incidence matrix for the above graph.
2. (10 pts.) Use Dijkstra’s algorithm to find a shortest path between A and Z. Label each node with the final L value of that node.
VIII. History and famous problems (10 pts.) Pick four of the following names of people and mathematical problems, and say something about them. Clearly indicate which person/problem you are describing (duplicates do not count if I have accidentally included one).

a) René Descartes  
b) Georg Cantor  
c) The 3x+1 conjecture  
d) Leonardo of Pisa  
e) Karl Friedrich Gauss  
f) Charles Dodgson  
g) Paul Gustav Heinrich Bachman  
h) Ada Augusta, Countess of Lovelace  
i) The halting problem  
j) Marin Mersenne  
k) James Bernoulli  
l) Pierre-Simon Laplace  
m) Abu Ja’far Mohammed Ibn Musa Al-Khowarizmi  
n) Donald Knuth  
o) George Boole  
p) Goldbach’s conjecture  
q) Pierre de Fermat  
r) G. Lejeune Dirichlet  
s) The twin primes conjecture  
t) Pierre-Simon Laplace  
u) Paul Erdős