Math 434 **Name :**
Final Exam (12-23,32)
May 10, 2004

Please put each problem on a separate sheet of paper. State all theorems you use.

1. (10)

a. Define what it means for a ring R to be an *integral domain.*

b. For what values of n is Z_n an integral domain?

2. (10) Let R be a ring and I be a *two-sided ideal.* Prove that R/I is commutative if and only if $rs - sr \in I$ for all $r, s \in R$.

3. (10) Let R and S be rings and $\phi : R \to S$ a *ring homomorphism.*

a. Prove that $Ker_\phi = \{x|\phi(x) = 0\}$ is an ideal of R

b. Prove that if $a \in R$ is an *idempotent*, then $\phi(a)$ is an *idempotent* of S

4. (10) Let F be a field, $a \in F$, and $f(x) \in F[x]$. Prove that a is a *zero of* $f(x)$ if and only if $x - a$ is a *factor* of $f(x)$

5. (10) Write the following polynomials as the product of irreducible factors over the given fields.

a. $3x^2 + x + 4$ over Z_7

b. $2x^2 + x + 3$ over Z_5

c. $x^5 + 3x + 3$ over Q

6. (10) Prove that a *Euclidean domain* is a *PID*

7. (10) Find the *splitting field* for $x^3 + x + 1$ over Z_2. Express $x^3 + x + 1$ as a product of linear factors over the splitting field.

8. (10) Find the *Galois group* of $Q\left(\sqrt{2}, \sqrt{5}\right)$ over Q. Find the fixed fields for each of the subgroups of the Galois group.

9. (10) Let R be a *finite, simple, commutative ring*. Prove that R is a *field* or R has a *prime number* of elements and $ab = 0$ for all $a, b \in R$.

10. (10) Let $p(x) = x^9 - 1 = (x^3 - 1)(x^6 + x^3 + 1) = (x - 1)(x^2 + x + 1)(x^6 + x^3 + 1)$ and let ω be a *primitive* 9*th root of unity*.

a. Prove that ω^3 is **not** a root of $(x^6 + x^3 + 1)$.

b. Prove that ω^3 and ω^6 are roots of $(x^2 + x + 1)$

c. Prove that ω is a root of $(x^6 + x^3 + 1)$

d. Find all the roots of $(x^6 + x^3 + 1)$

e. Find the Galois group of $x^6 + x^3 + 1$ over Q.