Population models

1. Consider a population that has a constant per capita rate of change in its natural state. Suppose humans harvest this population at a constant rate. (As an example, you might think of fishing regulations that allow a fixed number of fish to be caught each year.)

Let \( p(t) \) give the population at time \( t \). Let \( r \) be the constant per capita rate of change so in the natural state, we have

\[
\frac{p'(t)}{p(t)} = r \quad \text{or} \quad p'(t) = rp(t).
\]

Let \( h \) be the constant harvest rate. This is a negative contribution to the overall rate of change \( p'(t) \) so with harvesting included, we have

\[
p'(t) = rp(t) - h \quad \text{or} \quad \frac{dp}{dt} = rp - h.
\]

To solve this, first do some algebra to separate variables giving

\[
\frac{1}{rp - h} dp = dt
\]

so

\[
\int \frac{1}{rp - h} dp = \int dt.
\]

The right side is easy to evaluate. For the left side, factor \( r \) from the denominator giving

\[
\int \frac{1}{rp - h} dp = \frac{1}{r} \int \frac{1}{p - \frac{h}{r}} dp = \frac{1}{r} \ln \left| p - \frac{h}{r} \right|.
\]

We thus have

\[
\frac{1}{r} \ln \left| p - \frac{h}{r} \right| = t + C.
\]

Now we want to solve for \( p \). First, multiply both sides by \( r \) to get

\[
\ln \left| p - \frac{h}{r} \right| = rt + C
\]

where we rename \( rC \) as \( C \). This is equivalent to the exponential form

\[
\left| p - \frac{h}{r} \right| = e^{rt + C} = e^{rt}e^C = Ce^{rt}
\]

where we rename \( e^C \) as \( C \) in the last step. We can drop the absolute value by absorbing \( \pm \) into the constant \( C \) giving

\[
p - \frac{h}{r} = Ce^{rt}
\]
Thus

\[ p(t) = Ce^{rt} + \frac{h}{r}. \]

We can now solve for the constant \( C \) in terms of the initial population \( p_0 \) by evaluating this for \( t = 0 \). This gives

\[ p(0) = Ce^0 + \frac{h}{r} = C + \frac{h}{r} \]
to be compared with

\[ p(0) = p_0. \]

We must thus have

\[ C + \frac{h}{r} = p_0 \quad \text{or} \quad C = p_0 - \frac{h}{r}. \]

Substituting this into the general solution gives us

\[ p(t) = \left(p_0 - \frac{h}{r}\right)e^{rt} + \frac{h}{r}. \]

We can see that this is consistent with what we saw using the slope field. If \( p_0 = \frac{h}{r} \), the first term is zero and \( p(t) = \frac{h}{r} \) for all \( t \). If \( p_0 > \frac{h}{r} \), the first term is positive and gives exponential growth. If \( p_0 < \frac{h}{r} \), the first term is negative and results in a population of zero in finite time. This time \( T \) is found by solving

\[ 0 = \left(p_0 - \frac{h}{r}\right)e^{rT} + \frac{h}{r}. \]

The result is

\[ T = \frac{1}{r} \ln\left(\frac{h}{h - rp_0}\right). \]

2. Consider a population model in which the per capita rate of change is proportional to the difference between a constant \( K \) and the population. (The constant \( K \) is usually called the carrying capacity of the environment.)

We’ll just set up the differential equation here. Solving it involves some messy algebra. We’ll do this in class on Monday. Don’t worry about it for the exam.

Let \( p(t) \) be the population at time \( t \). The difference between \( K \) and the population is given by \( K - p(t) \). If the per capita rate of change is proportional to this difference, we can write

\[ \frac{p'(t)}{p(t)} = a(K - p(t)) \]

where \( a \) is the proportionality constant. Thus,

\[ p'(t) = ap(t)(K - p(t)) \quad \text{or} \quad \frac{dp}{dt} = ap(K - p). \]

To draw a slope field, note that the right side is zero for both \( p = 0 \) and \( p = K \). Thus, we will have horizontal tangent line segments at both \( p = 0 \) and \( p = K \).