Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. Only write on one side of each page.

“The shortest path between two truths in the real domain passes through the complex domain.” – Jacques Hadamard

“Such is the advantage of a well-constructed language that its simplified notation often becomes the source of profound theories.” – P. S. Laplace

“The road to wisdom? Well it plain and simple to express: Err and err and err again, but less and less and less.” - Piet Hein, poet and scientist (1905-1996)

Problems

1. Determine the maximal ideals of \( \mathbb{R}[x]/(x^2 - 3x + 2) \) where \( \mathbb{R} \) denotes the real numbers.

2. Prove either of the following:
   (a) \( \mathbb{Z}_2[x]/(x^3 + x + 1) \) is a field.
   (b) \( \mathbb{Z}_3[x]/(x^3 + x + 1) \) is not a field.

3. Adapt Euclid’s proof of the infinitude of prime integers to show that for any field \( F \), there are infinitely many monic irreducible polynomials in \( F[x] \).
   (a) Also explain why this argument fails for the formal power series ring \( F[[x]] \).

4. Partial Fractions for polynomials
   (a) Prove that every rational function in \( \mathbb{C}[x] \) can be written as a sum of a polynomial and a linear combination of functions of the form \( 1/(x-a)^i \).
   (b) Find a basis for \( \mathbb{C}(x) \) as a vector space over \( \mathbb{C} \).

5. Let \( a \) and \( b \) be relatively prime integers. Prove there are integers \( m, n \) such that \( a^m + b^n \equiv 1 \pmod{ab} \)