Due April 25

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. Only write on one side of each page.

“The road to wisdom? Well it plain and simple to express: Err and err and err again, but less and less and less.” -Piet Hein, poet and scientist (1905-1996)

Problems

1. Do both of the following
   (a) Judson Ch 19, #4: Let $B$ be the set of positive integers that are divisors of 36. Define an order on $B$ by $a \preceq b$ if $a|b$. Prove that $B$ is a Boolean algebra. Find a set $X$ such that $B$ is isomorphic to $P(X)$.
   (b) Judson Ch 19, #5: Prove or disprove: $\mathbb{Z}$ is a poset under the relation $a \preceq b$ if $a|b$.

2. Do both of the following
   (a) Judson Ch 19, #15: Let $R$ be a ring and suppose that $X$ is the set of ideals of $R$. Show that $X$ is a poset ordered by set-theoretic inclusion, $\subseteq$. Define the meet of two ideals $I$ and $J$ in $X$ by $I \cap J$ and the join of $I$ and $J$ by $I + J$. Prove that the set of ideals of $R$ is a lattice under these operations.
   (b) Judson Ch 19, #20: Let $X$ and $Y$ be posets. A map $\phi : X \rightarrow Y$ is order-preserving if $a \preceq b$ implies that $\phi(a)\preceq\phi(b)$. Let $L$ and $M$ be lattices. A map $\psi : L \rightarrow M$ is a lattice homomorphism if $\psi(a \lor b) = \psi(a) \lor \psi(b)$ and $\psi(a \land b) = \psi(a) \land \psi(b)$. Show that every lattice homomorphism is order-preserving, but that it is not the case that every order-preserving homomorphism is a lattice homomorphism.

3. Do both of the following
   (a) Judson Ch 19, #21: Let $B$ be a Boolean algebra. Prove $a = b$ if and only if $(a \land b') \lor (a' \land b) = O$ for $a, b \in B$.
   (b) Judson Ch 19, #22: Let $B$ be a Boolean algebra. Prove $a = 0$ if and only if $(a \land b') \lor (a' \land b) = b$ for all $b \in B$.

4. Judson Ch 19, #16: Let $B$ be a Boolean algebra. Prove each of the following identities.
   (a) $a \lor I = I$ and $a \land O = O$ for all $a \in B$.
   (b) If $a \lor b = I$ and $a \land b = O$, then $b = a'$.
   (c) $(a')' = a$ for all $a \in B$.
   (d) $I' = O$ and $O' = I$.
   (e) $(a \lor b)' = a' \land b'$ and $(a \land b)' = a' \lor b'$ (De Morgan’s laws).

5. Judson Ch 19, #10: (See page 321 of the textbook for the pictures of the three circuits.)
   For each of the following circuits, write a Boolean expression. If the circuit can be replaced by one with fewer switches, give the Boolean expression and draw a diagram for the new circuit.
6. Do both of the following:

(a) Let $F$ be a field. Find all elements $a$ such that $a = a^{-1}$

(b) Let $R$ be an integral domain containing a field $F$ as subring and which is finite-dimensional when viewed as a vector space over $F$. Prove that $R$ is a field.

7. Let $F$ be a field containing exactly 8 elements. Prove or disprove that the characteristic of $F$ is 2.

8. Let $\alpha$ be the real cube root of 2. What is the irreducible polynomial for $1 + \alpha^2$ over $\mathbb{Q}$?

9. Determine the irreducible polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over each of the following fields:

(a) $\mathbb{Q}$

(b) $\mathbb{Q}(\sqrt{5})$

(c) $\mathbb{Q}(\sqrt{10})$

(d) $\mathbb{Q}(\sqrt{15})$

10. Let $\alpha$ be a complex root of the irreducible (in $\mathbb{Q}$) polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in $F(\alpha)$. Write your answer in the form $a + b\alpha + c\alpha^2$ where $a, b, c \in \mathbb{Q}$. 