Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. Only write on one side of each page.

“Experience is what enables you to recognize a mistake when you make it again.” (Earl Wilson)

Problems

1. Given a ring $R$, the set of formal power series $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ (‘formal’ means there is no requirement of convergence) is a ring. (This set is denoted $R[[t]]$.) Show that $R[[t]]$ is a ring under the standard addition and multiplication of power series and prove that a formal power series $p(t)$ is invertible in $R[[t]]$ if and only if $a_0$ is a unit of $R$.

2. Let $\mathbb{Q}$ denote the rational numbers (you may use the fact that $\mathbb{Q}$ is a field), $\mathbb{Q}[\alpha]$ the smallest subring of $\mathbb{C}$ (the complex numbers) containing $\mathbb{Q}$ and $\alpha$, and $\mathbb{Q}[\alpha, \beta]$ the smallest subring of $\mathbb{C}$ containing $\mathbb{Q}$ and both $\alpha$ and $\beta$. Let $\alpha = \sqrt{2}$, $\beta = \sqrt{3}$ and $\gamma = \alpha + \beta$. Prove that $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$.

3. (Judson: Chapter 16, #6) Find all the ring homomorphisms $\phi: \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$.

4. Do both of the following:
   
   (a) Judson: Chapter 16, #36 (c),(d). You may assume parts (a) and (b) are true.
   
   (b) Let $I$ and $J$ be subrings of a ring $R$ and define $I + J = \{ a + b : a \in I \text{ and } b \in J \}$.
      
      i. Prove that $I + J$ need not be a subring of $R$.
      
      ii. Prove that if one of $I$ and $J$, say $J$, is an ideal, then $I + J$ is a subring of $R$. (Hence the Second Isomorphism Theorem for Rings in Judson [Theorem 16.13] now makes sense.)