More on equations of planes

So far, we have seen several forms for the equation of a plane:

\[ Ax + By + Cz + D = 0 \]  
standard form  

\[ z = m_x x + m_y y + b \]  
slopes-intercept form  

\[ z - z_0 = m_x (x - x_0) + m_y (y - y_0) \]  
point-slopes form

Using vectors, we can add another form that is coordinate-free.

A plane can be specified by giving a vector \( \mathbf{n} \) perpendicular to the plane (called a normal vector) and a point \( P_0 \) on the plane. We can develop a condition or test to determine whether or not a variable point \( P \) is on the plane by thinking geometrically and using the dot product. Here’s the reasoning:

- \( P \) is on the plane if and only if the vector \( \overrightarrow{P_0P} \) is parallel to the plane.
- The vector \( \overrightarrow{P_0P} \) is parallel to the plane if and only if \( \overrightarrow{P_0P} \) is perpendicular to the normal vector \( \mathbf{n} \).
- The vectors \( \overrightarrow{P_0P} \) and \( \mathbf{n} \) are perpendicular if and only if their dot product is zero:

\[ \mathbf{n} \cdot \overrightarrow{P_0P} = 0. \]

So, the condition \( \mathbf{n} \cdot \overrightarrow{P_0P} = 0 \) is a new form for the equation of a plane. We’ll refer to this as the point-normal form. We can see how the point-normal form relates to our familiar forms by introducing coordinates and components. Let \( P_0 \) have coordinates \((x_0, y_0, z_0)\), the variable point \( P \) have coordinates \((x, y, z)\), and the normal vector \( \mathbf{n} \) have components \( (n_x, n_y, n_z) \). With these, the vector \( \overrightarrow{P_0P} \) has components \( (x - x_0, y - y_0, z - z_0) \). So, the point-normal form can be written as

\[ 0 = \mathbf{n} \cdot \overrightarrow{P_0P} \]

\[ = (n_x, n_y, n_z) \cdot (x - x_0, y - y_0, z - z_0) \]

\[ = n_x (x - x_0) + n_y (y - y_0) + n_z (z - z_0) \]

\[ = n_x x + n_y y + n_z z - (n_x x_0 + n_y y_0 + n_z z_0). \]

The last expression is the same as \( Ax + By + Cz + D \) if we identify \( n_z \) as \( A \), \( n_y \) as \( B \), \( n_z \) as \( C \) and \( -(n_x x_0 + n_y y_0 + n_z z_0) \) as \( D \). This is perhaps easier to see in an example.

**Example**

Find the standard form for the equation of the plane that contains the point \((6, 5, 2)\) and has normal vector \((7, -3, 4)\).

With \((x, y, z)\) as the coordinates of a variable point, we can write

\[ 0 = \mathbf{n} \cdot \overrightarrow{P_0P} \]

\[ = (7, -3, 4) \cdot (x - 6, y - 5, z - 2) \]

\[ = 7(x - 6) - 3(y - 5) + 4(z - 2) \]

\[ = 7x - 3y + 4z - 42 + 15 - 8 \]

\[ = 7x - 3y + 4z - 35. \]

So the standard form of the equation for this plane is \(7x - 3y + 4z - 35 = 0\).
Exercises

1. Use the point-normal equation to determine which, if any, of the following points are on the plane that has normal vector \(2\hat{i} - \hat{j} + 6\hat{k}\) and contains the point \((3, 4, 2)\).
   
   (a) \((5, -4, 0)\)  
   (b) \((1, 6, 2)\)  
   (c) \((2, 8, 3)\)

   \textit{Answer:} \((5, -4, 0)\) and \((2, 8, 3)\) are on the plane, \((1, 6, 2)\) is not

2. Find the slopes-intercept form of the equation that contains the point \((4, 2, -7)\) and has normal vector \(\vec{n} = 5\hat{i} - 3\hat{j} + 2\hat{k}\).

   \textit{Answer:} \(z = -\frac{5}{2}x + \frac{3}{2}y\)

3. Find the slopes-intercept form of the equation for the plane that contains the point \((4, 2, -7)\) and has normal vector \(\vec{n} = \langle -6, 1, 5 \rangle\).

   \textit{Answer:} \(z = \frac{6}{5}x - \frac{1}{5}y - \frac{57}{5}\)

4. Find the standard form of the equation for the plane that contains the point \((6, 3, 0)\) and is parallel to a second plane given by the equation \(5x + 2y - 9z = 14\).

5. Find the standard form of the equation for the plane that contains the point \((7, -2, 1)\) and is perpendicular to the vector from the origin to that same point.

   \textit{Answer:} \(7x - 2y + z - 54 = 0\)