Methods of Proof

Direct Proof of $H \implies C$
1. Start with the hypotheses of $H$.
2. Use nothing but allowable justifications.
3. Deduce $C$.

Direct Proof of $(H_1 \land \cdots \land H_n) \implies C$
1. List all of the individual hypotheses $H_1, \cdots, H_n$ as given.
2. Use nothing but allowable justifications.
3. Deduce $C$.

Direct Proof of $H \implies (C_1 \land \cdots \land C_n)$ [Note this is equivalent to] $((H \implies C_1) \land \cdots \land (H \implies C_n))$
1. Start with the hypotheses of $H$.
2. Use nothing but allowable justifications.
3. Deduce $C_1$.
4. Independently deduce each of the remaining $C_i$.

Use of the Contrapositive to prove $H \implies C$
1. Present a direct proof of $\neg C \implies \neg H$.
2. That is, Start with $\neg C$
3. Use nothing but logical steps
4. Deduce $\neg H$

Proof by Contradiction of $H \implies C$ (Not liked by Constructivists)
1. Start with the hypothesis $H$.
2. Suppose the RAA hypothsis $(\neg C)$
3. Use $H$, $(\neg C)$ and nothing but allowable justifications to deduce $(D \land (\neg D))$
4. Conclude $\neg (\neg C)$

How to deal with disjunctions

Disjoined Hypotheses $H_1 \lor \cdots \lor H_n \implies C$
1. Do it by cases: Solve the $n$ individual problems $H_1 \implies C$, $H_2 \implies C, \cdots, H_n \implies C$
Disjoined Conclusions \( H \implies C_1 \lor \cdots \lor C_n \)

1. Note: If any \( C_i \) can be deduced from \( H \) then the result is true.
2. Method of “hidden cases”: Start with \( C \) and the negation of all but one \( C_i \)
3. Deduce this last \( C_i \).

**How to prove Universal statements** \( \forall x \ (p(x) \implies q(x)) \)

1. Start with an arbitrary element \( x \) in the universal set \( X \)
2. Show that \( p(x) \implies q(x) \) using only the properties of \( x \) that make it an element of \( X \),

**How to prove Existential Statements** \( \exists x \ p(x) \)

1. Best approach is to actually exhibit an instance of \( x \) for which \( p(x) \) is true.
2. If the above doesn’t work, try a proof by contradiction.
   (a) Suppose for every \( x \), \( p(x) \) fails to be true and arrive at a contradiction.

**Forward-Backward method for doing proofs**
- Write out the hypotheses and the conclusions with space between
- Alternate between
  1. Logically deducing facts from the hypotheses
  2. Determining facts that imply the conclusions
  3. Join in the middle

**Allowable Rules of Inference (deductions): these are all tautologies**

1. **Modus Ponens** (mode that affirms) \( ((p \implies q) \land p) \implies q \)
2. **Syllogism** \( ((p \implies q) \land (q \implies r)) \implies (p \implies r) \)
3. **Modus Tollens** (mode that denies) \( ((p \implies q) \land (\neg q)) \implies (\neg p) \)
4. **Contradiction** \( ((p \land (\neg q)) \implies (r \land (\neg r))) \implies q \)
5. Tautology affirming using the contrapositive is valid: \( (p \implies q) \iff ((\neg q) \implies (\neg p)) \)

**Terminology**
- \( ((\neg q) \implies (\neg p)) \) is called the **contrapositive** of \( (p \implies q) \)
- \( (q \implies p) \) is called the **converse** of \( (p \implies q) \)
- \( (\neg p \implies \neg q) \) is called the **obverse** of \( (p \implies q) \)