Directions:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Examination 5

Do Two (2) of these “Computational” Problems

C.1. Without using technology, compute the determinant of the matrix

\[
\begin{bmatrix}
0 & -1 & 0 & 1 \\
-2 & 3 & 1 & 6 \\
1 & -2 & 2 & 3 \\
0 & 1 & 0 & -2
\end{bmatrix}
\]

C.2. Prove that the function \( T : M_{n,n} \rightarrow M_{n,n} \) given by \( T(A) = A + A^t \) is a linear transformation.

C.3. The number \( \lambda = 2 \) is an eigenvalue of the matrix

\[
\begin{bmatrix}
3 & -2 & 2 \\
-4 & 1 & -2 \\
-5 & 1 & -2
\end{bmatrix}
\]

Determine a basis for the eigenspace, \( E_A(2) \), corresponding to this eigenvalue and state the geometric multiplicity \( \gamma_A(2) \) of this eigenvalue.

Do Two (2) of these “In text, class or homework” problems

M.1. Prove two (2) of the following.

(a) If \( A \) is diagonalizable and \( B \) is similar to \( A \) then \( B \) is diagonalizable.

(b) If \( A \) is diagonalizable and invertible then \( A^{-1} \) is diagonalizable.

(c) Suppose \( A \) and \( B \) have the same eigenvalues and each eigenvalue has the same algebraic and geometric multiplicity in \( A \) as it does in \( B \). If \( A \) is diagonalizable, then \( A \) and \( B \) are similar.

M.2. A square matrix \( A \) is \textbf{idempotent} if \( A^2 = A \). Show that if \( A \) is an idempotent matrix then the numbers 0 and 1 are both eigenvalues of \( A \) and that they are the only eigenvalues of \( A \).
M.3. Theorem ILTLI (Injective Linear Transformations and Linear Independence) tells us that if $T : U \to V$ is a linear transformation then the image of any linearly independent set is linearly independent. Without using this theorem, prove that if $S = \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \}$ is a linearly independent set in the vector space $U$ and $T : U \to V$ is an injective linear transformation, then $R = \{ T(\vec{u}_1), T(\vec{u}_2), T(\vec{u}_3) \}$ is a linearly independent set in the vector space $V$.

Do two (2) of these “Other” problems

T.1. The set $B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{C}^2$. Define a function $T : \mathbb{C}^2 \to \mathbb{C}^2$ by: if $\vec{x} = a \begin{bmatrix} 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then $T(\vec{x}) = a \begin{bmatrix} 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Use the fact (which you do not have to prove) that $T$ is a linear transformation to find the matrix $A$ that satisfies $T(\vec{x}) = A\vec{x}$ for every vector $\vec{x} \in \mathbb{C}^2$.

T.2. Suppose that $A$ is a $4 \times 4$ matrix with exactly two distinct eigenvalues, 6 and $-7$ and let $E_A(6)$ and $E_A(-7)$ be the respective eigenspaces.

(a) Write all possible characteristic polynomials of $A$ that are consistent with $E_A(6) = 3$

(b) Write all possible characteristic polynomials of $A$ that are consistent with $E_A(-7) = 2$.

T.3. An $n \times n$ matrix $A$ is called nilpotent if, for some positive integer $k$, $A^k = O$, where $O$ is the $n \times n$ zero matrix. Prove that 0 is the only eigenvalue of any nilpotent matrix.