Definitions, Axioms, Postulates, Propositions, and Theorems from *Euclidean and Non-Euclidean Geometries* by Marvin Jay Greenberg

**Undefined Terms:** Point, Line, Incident, Between, Congruent.

**Incidence Axioms:**

IA1: For every two distinct points there exists a unique line incident on them.

IA2: For every line there exist at least two points incident on it.

IA3: There exist three distinct points such that no line is incident on all three.

**Incidence Propositions:**

P2.1: If \(l\) and \(m\) are distinct lines that are non-parallel, then \(l\) and \(m\) have a unique point in common.

P2.2: There exist three distinct lines such that no point lies on all three.

P2.3: For every line there is at least one point not lying on it.

P2.4: For every point there is at least one line not passing through it.

P2.5: For every point there exist at least two distinct lines that pass through it.

**Betweenness Axioms:**

B1: If \(A \ast B \ast C\), then \(A\), \(B\), and \(C\) are three distinct points all lying on the same line, and \(C \ast B \ast A\).

B2: Given any two distinct points \(B\) and \(D\), there exist points \(A\), \(C\), and \(E\) lying on \(\overrightarrow{BD}\) such that \(A \ast B \ast D\), \(B \ast C \ast D\), and \(B \ast D \ast E\).

B3: If \(A\), \(B\), and \(C\) are three distinct points lying on the same line, then one and only one of them is between the other two.

B4: For every line \(l\) and for any three points \(A\), \(B\), and \(C\) not lying on \(l\):

1. If \(A\) and \(B\) are on the same side of \(l\), and \(B\) and \(C\) are on the same side of \(l\), then \(A\) and \(C\) are on the same side of \(l\).
2. If \(A\) and \(B\) are on opposite sides of \(l\), and \(B\) and \(C\) are on opposite sides of \(l\), then \(A\) and \(C\) are on the same side of \(l\).

Corollary: If \(A\) and \(B\) are on opposite sides of \(l\), and \(C\) is on the same side of \(l\), then \(A\) and \(C\) are on opposite sides of \(l\).

**Betweenness Definitions:**

- **Segment** \(AB\): Point \(A\), point \(B\), and all points \(P\) such that \(A \ast P \ast B\).
- **Ray** \(\overrightarrow{AB}\): Segment \(AB\) and all points \(C\) such that \(A \ast B \ast C\).
- **Line** \(\overline{AB}\): Ray \(\overrightarrow{AB}\) and all points \(D\) such that \(D \ast A \ast B\).

**Same/Opposite Side:** Let \(l\) be any line, \(A\) and \(B\) any points that do not lie on \(l\). If \(A = B\) or if segment \(AB\) contains no point lying on \(l\), we say \(A\) and \(B\) are on the same side of \(l\), whereas if \(A \neq B\) and segment \(AB\) does intersect \(l\), we say that \(A\) and \(B\) are on opposite sides of \(l\). The law of excluded middle tells us that \(A\) and \(B\) are either on the same side or on opposite sides of \(l\).

**Betweenness Propositions:**

P3.1: For any two points \(A\) and \(B\):

1. \(\overrightarrow{AB} \cap \overrightarrow{BA} = AB\), and
2. \(\overrightarrow{AB} \cup \overrightarrow{BA} = \overrightarrow{AB}\).

P3.2: Every line bounds exactly two half-planes and these half-planes have no point in common.
**Pasch’s Theorem:** If A, B, and C are distinct points and l is any line intersecting AB in a point between A and B, then l also intersects either AC or BC. If C does not lie on l, then l does not intersect both AC and BC.

**Interior of a Triangle:** The interior of a triangle is the intersection of the interiors of its three angles. Define a point to be exterior to the triangle if it is in not in the interior and does not lie on any side of the triangle.

**Angle Propositions:**

**Problem 9**: Given a line l, a point A on l and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B.

**Crossbar Theorem:** If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.

**Congruence Axioms:**

**C1:** If $A$ and $B$ are distinct points and if $A'$ is any point, then for each ray $r$ emanating from $A'$ there is a unique point $B'$ on $r$ such that $B' \neq A'$ and $AB \cong A'B'$.

**C2:** If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

**C3:** If $A\ast B\ast C$, and $A'\ast B'\ast C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

**C4:** Given any $\angle BAC$ (where by definition of angle, $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$ and is distinct from $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point $A'$, then there is a unique ray $\overrightarrow{A'C'}$ on a given side of line $\overrightarrow{A'B'}$ such that $\angle B'A'C' \cong \angle BAC$.

**C5:** If $\angle A \cong \angle B$ and $\angle A \cong \angle C$, then $\angle B \cong \angle C$. Moreover, every angle is congruent to itself.

**C6 (SAS):** If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.
Congruence Propositions:

P3.10: If in $\triangle ABC$ we have $AB \cong AC$, then $\angle B \cong \angle C$.

P3.11: If $A*B*C$, $D*E*F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

P3.12: Given $AC \cong DF$, then for any point $B$ between $A$ and $C$, there is a unique point $E$ between $D$ and $F$ such that $AB \cong DE$.

P3.13: 1. Exactly one of the following holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
   2. If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
   3. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
   4. If $AB < CD$ and $CD < EF$, then $AB < EF$.

P3.14: Supplements of Congruent angles are congruent.

P3.15: 1. Vertical angles are congruent to each other.
   2. An angle congruent to a right angle is a right angle.

P3.16: For every line $l$ and every point $P$ there exists a line through $P$ perpendicular to $l$.

P3.17 (ASA): Given $\triangle ABC$ and $\triangle DEF$ with $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.18: In $\triangle ABC$ we have $\angle B \cong \angle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

P3.19: Given $\overline{BC}$ between $\overline{BA}$ and $\overline{BC}$, $\overline{EH}$ between $\overline{ED}$ and $\overline{EF}$, $\angle CBG \cong \angle FEH$ and $\angle GBA \cong \angle HED$. Then $\angle ABC \cong \angle DEF$.

P3.20: Given $\overline{BC}$ between $\overline{BA}$ and $\overline{BC}$, $\overline{EH}$ between $\overline{ED}$ and $\overline{EF}$, $\angle CBG \cong \angle FEH$ and $\angle ABC \cong \angle DEF$. Then $\angle GBA \cong \angle HED$.

P3.21: 1. Exactly one of the following holds: $\angle P < \angle Q$, $\angle P \cong \angle Q$, or $\angle P > \angle Q$.
   2. If $\angle P < \angle Q$ and $\angle Q \cong \angle R$, then $\angle P < \angle R$.
   3. If $\angle P > \angle Q$ and $\angle Q \cong \angle R$, then $\angle P > \angle R$.
   4. If $\angle P < \angle Q$ and $\angle Q < \angle R$, then $\angle P < \angle R$.

P3.22 (SSS): Given $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.23: All right angles are congruent to each other.

Corollary (not numbered in text) If $P$ lies on $l$ then the perpendicular to $l$ through $P$ is unique.

Definitions:

Segment Inequality: $AB < CD$ (or $CD > AB$) means that there exists a point $E$ between $C$ and $D$ such that $AB \cong CE$.

Angle Inequality: $\angle ABC < \angle DEF$ means there is a ray $\overline{EG}$ between $\overline{ED}$ and $\overline{EF}$ such that $\angle ABC \cong \angle GEF$.

Right Angle: An angle $\angle ABC$ is a right angle if has a supplementary angle to which it is congruent.

Parallel: Two lines $l$ and $m$ are parallel if they do not intersect, i.e., if no point lies on both of them.

Perpendicular: Two lines $l$ and $m$ are perpendicular if they intersect at a point $A$ and if there is a ray $\overline{AB}$ that is a part of $l$ and a ray $\overline{AC}$ that is a part of $m$ such that $\angle BAC$ is a right angle.

Triangle Congruence and Similarity: Two triangles are congruent if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. Similar triangles have this one-to-one correspondence only with their angles.

Circle (with center $O$ and radius $OA$): The set of all points $P$ such that $OP$ is congruent to $OA$.

Triangle: The set of three distinct segments defined by three non-collinear points.
Continuity Axioms:

**Archimedes’ Axiom:** If $AB$ and $CD$ are any segments, then there is a number $n$ such that if segment $CD$ is laid off $n$ times on the ray $\overrightarrow{AB}$ emanating from $A$, then a point $E$ is reached where $n \cdot CD \cong AE$ and $B$ is between $A$ and $E$.

**Dedekind’s Axiom:** Suppose that the set of all points on a line $l$ is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of $\Sigma_1$ is between two points of $\Sigma_2$ and visa versa. Then there is a unique point $O$ lying on $l$ such that $P_1 \ast O \ast P_2$ if and only if one of $P_1$, $P_2$ is in $\Sigma_1$, the other in $\Sigma_2$ and $O \neq P_1, P_2$. A pair of subsets $\Sigma_1$ and $\Sigma_2$ with the properties in this axiom is called a Dedekind cut of the line $l$.

**Continuity Principles:**

**Circular Continuity Principle:** If a circle $\gamma$ has one point inside and one point outside another circle $\gamma'$, then the two circles intersect in two points.

**Elementary Continuity Principle:** In one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

Other Theorems, Propositions, and Corollaries in Neutral Geometry:

**T4.1:** If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

**Corollary 1:** Two lines perpendicular to the same line are parallel. Hence the perpendicular dropped from a point $P$ not on line $l$ to $l$ is unique.

**Corollary 2:** If $l$ is any line and $P$ is any point not on $l$, there exists at least one line $m$ through $P$ parallel to $l$.

**T4.2 (Exterior Angle Theorem):** An exterior angle of a triangle is greater than either remote interior angle.

**T4.3 (see text for details):** There is a unique way of assigning a degree measure to each angle, and, given a segment $OI$, called a unit segment, there is a unique way of assigning a length to each segment $AB$ that satisfies our standard uses of angle and length.

**Corollary 1:** The sum of the degree measures of any two angles of a triangle is less than $180^\circ$.

**Corollary 2:** If $A$, $B$, and $C$ are three noncollinear points, then $\overrightarrow{AC} < \overrightarrow{AB} + \overrightarrow{BC}$.

**T4.4 (Saccheri-Legendre):** The sum of the degree measures of the three angles in any triangle is less than or equal to $180^\circ$.

**Corollary 1:** The sum of the degree measures of two angles in a triangle is less than or equal to the degree measure of their remote exterior angle.

**Corollary 2:** The sum of the degree measures of the angles in any convex quadrilateral is at most $360^\circ$ (note: quadrilateral $\square ABCD$ is convex if it has a pair of opposite sides such that each is contained in a half-plane bounded by the other.)

**P4.1 (SAA):** Given $AC \cong DF$, $\angle A \cong \angle D$, and $\angle B \cong \angle E$. Then $\triangle ABC \cong \triangle DEF$.

**P4.2:** Two right triangles are congruent if the hypotenuse and leg of one are congruent respectively to the hypotenuse and a leg of the other.

**P4.3:** Every segment has a unique midpoint.

**P4.4:**

1. Every angle has a unique bisector.
2. Every segment has a unique perpendicular bisector.

**P4.5:** In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\angle C > \angle A$.

**P4.6:** Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\angle B < \angle B'$ if and only if $AC < A'C'$. 

4
Note: Statements up to this point are from or form neutral geometry. Choosing Hilbert’s/Euclid’s Axiom (the two are logically equivalent) or the Hyperbolic Axiom will make the geometry Euclidean or Hyperbolic, respectively.

Parallelism Axioms:

**Hilbert’s Parallelism Axiom for Euclidean Geometry:** For every line \( l \) and every point \( P \) not lying on \( l \) there is at most one line \( m \) through \( P \) such that \( m \) is parallel to \( l \). (Note: it can be proved from the previous axioms that, assuming this axiom, there is EXACTLY one line \( m \) parallel to \( l \) [see T4.1 Corollary 2]).

**Euclid’s Fifth Postulate:** If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

**Hyperbolic Parallel Axiom:** There exist a line \( l \) and a point \( P \) not on \( l \) such that at least two distinct lines parallel to \( l \) pass through \( P \).

Hilbert’s Parallel Postulate is logically equivalent to the following:

**T4.5:** Euclid’s Fifth Postulate.
**P4.7:** If a line intersects one of two parallel lines, then it also intersects the other.
**P4.8:** Converse to Theorem 4.1.
**P4.9:** If \( t \) is transversal to \( l \) and \( m \parallel l \), and \( t \perp l \), then \( t \perp m \).
**P4.10:** If \( k \parallel l \), \( m \perp k \), and \( n \perp l \), then either \( m = n \) or \( m \parallel n \).
**P4.11:** The angle sum of every triangle is 180°.

**Wallis:** Given any triangle \( \triangle ABC \) and given any segment \( DE \). There exists a triangle \( \triangle DEF \) (having \( DE \) as one of its sides) that is similar to \( \triangle ABC \) (denoted \( \triangle DEF \sim \triangle ABC \)).

Theorems 4.6 and 4.7 (see text) are used to prove P4.11. They define the defect of a triangle to be the 180° minus the angle sum, then show that if one defective triangle exists, then all triangles are defective. Or, in contrapositive form, if one triangle has angle sum 180°, then so do all others. They do not assume a parallel postulate.

Theorems Using the Parallel Axiom

**Parallel Projection Theorem:** Given three parallel lines \( l, m, \) and \( n \). Let \( t \) and \( t' \) be transversals to these parallels, cutting them in points \( A, B, \) and \( C \) and in points \( A', B', \) and \( C' \), respectively. Then \( \frac{AB}{BC} = \frac{A'B'}{B'C'} \).

**Fundamental Theorem on Similar Triangles:** Given \( \triangle ABC \sim \triangle A'B'C' \). Then the corresponding sides are proportional.
HYPERBOLIC GEOMETRY

L6.1: There exists a triangle whose angle sum is less than 180°.

Universal Hyperbolic Theorem: In hyperbolic geometry, from every line \( l \) and every point \( P \) not on \( l \) there pass through \( P \) at least two distinct parallels to \( l \).

T6.1: Rectangles do not exist and all triangles have angle sum less than 180°.

Corollary: In hyperbolic geometry, all convex quadrilaterals have angle sum less than 360°.

T6.2: If two triangles are similar, they are congruent.

T6.3: If two lines through \( O \) are situated symmetrically about \( l \) and \( l' \) are any distinct parallel lines, then any set of points on \( l \) equidistant from \( l' \) has at most two points in it.

T6.4: If two lines through \( O \) are parallel lines for which there exists a pair of points \( A \) and \( B \) on \( l \) equidistant from \( l' \), then \( l \) and \( l' \) have a common perpendicular segment that is also the shortest segment between \( l \) and \( l' \).

T6.5: If lines \( l \) and \( l' \) have a common perpendicular \( MM' \), then they are parallel and \( MM' \) is unique. Moreover, if \( A \) and \( B \) are points on \( l \) such that \( M \) is the midpoint of segment \( AB \), then \( A \) and \( B \) are equidistant from \( l' \).

T6.6: For every line \( l \) and every point \( P \) not on \( l \), let \( Q \) be the foot of the perpendicular from \( P \) to \( l \). Then there are two unique rays \( \overrightarrow{PX} \) and \( \overrightarrow{PY} \) on opposite sides of \( P \overline{Q} \) that do not meet \( l \) and have the property that a ray emanating from \( P \) meets \( l \) if and only if it is between \( \overrightarrow{PX} \) and \( \overrightarrow{PY} \). Moreover, these limiting rays are situated symmetrically about \( P \overline{Q} \) in the sense that \( \overrightarrow{xXPQ} \cong \overrightarrow{x'X'PQ} \).

T6.7: Given \( m \) parallel to \( l \) such that \( m \) does not contain a limiting parallel ray to \( l \) in either direction. Then there exists a common perpendicular to \( m \) and \( l \), which is unique.

Results from chapter 7 (Contextual definitions not included):

P7.1 1. \( P = P' \) if and only if \( P \) lies on the circle of inversion \( \gamma \).
       2. If \( P \) is inside \( \gamma \) then \( P' \) is outside \( \gamma \), and if \( P \) is outside \( \gamma \), then \( P' \) is inside \( \gamma \).
       3. \((P')' = P\).

P7.2 Suppose \( P \) is inside \( \gamma \). Let \( TU \) be the chord of \( \gamma \) which is perpendicular to \( \overrightarrow{OP} \). Then the inverse \( P' \) of \( P \) is the pole of chord \( TU \), i.e., the point of intersection of the tangents to \( \gamma \) at \( T \) and \( U \).

P7.3 If \( P \) is outside \( \gamma \), let \( Q \) be the midpoint of segment \( OP \). Let \( \sigma \) be the circle with center \( Q \) and radius \( \overrightarrow{OQ} = \overrightarrow{QP} \). Then \( \sigma \) cuts \( \gamma \) in two points \( T \) and \( U \), \( \overrightarrow{PT} \) and \( \overrightarrow{PU} \) are tangent to \( \gamma \), and the inverse \( P' \) of \( P \) is the intersection of \( TU \) and \( OP \).

P7.4 Let \( T \) and \( U \) be points on \( \gamma \) that are not diametrically opposite and let \( P \) be the pole of \( TU \). Then \( PT \cong PU \), \( \overrightarrow{PTU} \cong \overrightarrow{PUT} \), \( \overrightarrow{OP} \perp \overrightarrow{TU} \), and the circle \( \delta \) with center \( P \) and radius \( \overrightarrow{PT} = \overrightarrow{PU} \) cuts \( \gamma \) orthogonally at \( T \) and \( U \).

L7.1 Given that point \( O \) does not lie on circle \( \delta \).
       1. If two lines through \( O \) intersect \( \delta \) in pairs of points \( (P_1, P_2) \) and \( (Q_1, Q_2) \), respectively, then we have \( (\overrightarrow{OP_1})(\overrightarrow{OP_2}) = (\overrightarrow{OQ_1})(\overrightarrow{OQ_2}) \). This common product is called the power of \( O \) with respect to \( \delta \) when \( O \) is outside of \( \delta \), and minus this number is called the power of \( O \) when \( O \) is inside \( \delta \).
       2. If \( O \) is outside \( \delta \) and a tangent to \( \delta \) from \( O \) touches \( \delta \) at point \( T \), then \((\overrightarrow{OT})^2 \) equals the power of \( O \) with respect to \( \delta \).

P7.5 Let \( P \) be any point which does not lie on circle \( \gamma \) and which does not coincide with the center \( O \) of \( \gamma \), and let \( \delta \) be a circle through \( P \). Then \( \delta \) cuts \( \gamma \) orthogonally if and only if \( \delta \) passes through the inverse point \( P' \) of \( P \) with respect to \( \gamma \).