I affirm this work abides by the university’s Academic Honesty Policy.

Directions:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Do two (2) of these "Computational" problems

C.1. [15 points] Using anything you know about determinants, compute the determinant of the following matrix by hand.

\[
\begin{bmatrix}
0 & 0 & 1 & -1 & -1 \\
2 & 4 & 2 & 4 & 2 \\
2 & 4 & 3 & 0 & 3 \\
3 & 6 & 6 & 3 & 6 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

C.2. [9, 6 points] Recall that the zero vector of the vector space \( F(C, C) = \{ f : C \rightarrow C \} \) is the function \( Z : C \rightarrow C \) defined by \( Z(x) = 0 \) for all \( x \in C \). Consider the span \( V = \langle \{ e^{kx} | k \in C \} \rangle \) which is a subspace of \( F(C, C) \).

1. (a) Prove that \( W = \{ f \in V | f'' - 3f' + f = Z \} \) is a subspace of \( V \).
   (b) Find a basis for \( W \).

C.3. [15 Points] Find a basis for the kernel of the linear transformation \( T : M_{2,2} \rightarrow M_{2,2} \) defined by \( T(A) = \frac{1}{2}A - \frac{1}{2}A^t \).

Do any two (2) of these "Similar to In Class, Text, or Homework" problems

M.1. [15 Points] Prove that if \( T : U \rightarrow V \) is a linear transformation and \( W \) is a subspace of \( U \) then the image of \( W \) under \( T \), \( T(W) = \{ T(\bar{u}) : \bar{u} \in W \} \), is a subspace of \( V \).

M.2. [15 Points] Prove Theorem SMEE (Similar Matrices have Equal Eigenvalues)

1. Suppose \( A \) and \( B \) are similar matrices. Then the characteristic polynomials of \( A \) and \( B \) are equal, that is, \( \rho_A(x) = \rho_B(x) \).

M.3. [15 Points] Prove Theorem EER Eigenvalues, Eigenvectors, Representations: Suppose that \( T : V \rightarrow V \) is a linear transformation and \( B = \{ u_1, u_2, \cdots , u_n \} \) is a basis of \( V \). Then \( v \in V \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \) if and only if \( \rho_B(v) \) is an eigenvector of \( M^T_{B,B} \) for the eigenvalue \( \lambda \).
Do two (2) of these "Other" problems

T.1. [15 Points] In Proof LT-1 you saw a one-step test for whether or not a function is a linear transformation. This problem gives a one-step test for showing a subset of a vector space is a subspace.

1. Prove that a subset \( W \) of a vector space \( V \) is a subspace if and only if \( \alpha \vec{w}_1 + \beta \vec{w}_2 \in W \) is true for all \( \vec{w}_1, \vec{w}_2 \in W \) and for all \( \alpha, \beta \in \mathbb{C} \).

T.2. [15 Points] Let \( B = \{ e^x, xe^x, x^2 e^x \} \) be a basis for the subspace \( V \) of the vector space of functions with domain and codomain the set of complex numbers: \( F(C, C) = \{ f \mid f : C \to C \} \)

1. (a) Find the matrix representation \( M_{B,B}^T \) of the linear transformation \( T : V \to V \) defined by \( T(f) = f' \).

   (b) Use this matrix representation to find the kernel of \( T \), \( \ker(T) \).

T.3. [15 Points] It is a true fact that if \( V = \{ A \in M_{n,n} \mid A \) is symmetric} \) and \( W = \{ B \in M_{n,n} \mid B \) is skew-symmetric} \) then \( M_{n,n} = V \oplus W \). Prove this fact in the special case when \( n = 2 \).

T.4. [15 points] Professor Beezer has proven that if \( V \) is a finite-dimensional vector space and \( T : V \to V \) has \( \text{Range}(T) = V \) then \( T \) is an isomorphism. Show that this is not necessarily the case if \( V \) is infinite dimensional by giving an example of a linear transformation \( T : P \to P \) that is not injective but that has \( \text{Range}(T) = P \). Be sure to explain why your example has the desired properties. [Recall that \( P \) is the infinite dimensional vector space of all polynomials.]

You must do both of these problems ON THIS SHEET

R.1. [15 points] Prove that the set \( V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 : 5x_1 - 7x_2 - 2x_3 = 0 \right\} \) is a subspace of \( \mathbb{C}^3 \) by applying the three-part test of Theorem TSS. Write your proof according to the standards of this semester’s writing exercises.

R.2. [15 points] Suppose that \( Z : V \to V \) is the linear transformation denoted by \( Z(\mathbf{v}) = \mathbf{0} \) for all \( \mathbf{v} \in V \) (i.e. \( Z \) is the “zero” linear transformation). Suppose that \( T : V \to V \) is a linear transformation such that \( T^4 = Z \) (where \( T^4 = T \circ T \circ T \circ T \)). Then prove that \( T \) is not invertible. Write your proof according to the standards of this semester’s writing exercises.