Directions:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Do two (2) of these “Computational” problems

C.1. [15 points] Solve the following system of equations by hand.

\[
\begin{align*}
x_3 - x_4 - x_5 &= 4 \\
2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 &= 4 \\
2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 &= 4 \\
3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 &= 6
\end{align*}
\]

C.2. [15 points] Let \( V \) be the vector space of all functions of the form \( f(t) = c_1 \cos(t) + c_2 \sin(t) \) where \( c_1 \) and \( c_2 \) are arbitrary complex numbers. That is, \( V \) is the subspace of \( F \) with basis \( B = \{ \cos(t), \sin(t) \} \).

1. Find the matrix representation \( M^T_{B,B} \) of the linear transformation \( T : V \to V \) defined by \( T(f) = f'' + 3f' + 2f \).

2. Is \( T \) an isomorphism?

C.3. [15 Points] Find a basis for the range of the linear transformation \( T : M_{2,2} \to M_{2,2} \) defined by \( T(A) = \frac{1}{2}A + \frac{1}{2}A^t \).

Do any two (2) of these ”Similar to In Class, Text, or Homework” problems

M.1. [15 Points] Prove that \( T : U \to V \) is a linear transformation if and only if \( T(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) = \alpha_1 T(\vec{u}_1) + \alpha_2 T(\vec{u}_2) \) for all \( \vec{u}_1, \vec{u}_2 \in U \) and all \( \alpha_1, \alpha_2 \in \mathbb{C} \).

M.2. [15 Points] Prove that if \( T : U \to V \) is a linear transformation and \( X \) is a subspace of \( V \) then the pre-image of \( X \) under \( T \), \( T^{-1}(X) = \{ \vec{u} \in U : T(\vec{u}) \in X \} \), is a subspace of \( U \).

M.3. [15 Points] Use matrix multiplication notation to prove that if \( A \in M_{mn} \) and \( B \in M_{np} \) then

1. \( N(B) \subseteq N(AB) \)
2. \( C(AB) \subseteq C(A) \)
Do two (2) of these "Other" problems

T.1. [15 Points] Given that \( A \in M_{mn} \) and \( B \in M_{nm} \) where \( m \neq n \) and \( AB = I_m \). Use a proof by contradiction to show that the columns of \( B \) must be linearly independent.

T.2. [15 Points] Let \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) and define a function \( T : P_2 \rightarrow M_{2,2} \) by \( T(p) = p(A) \).

1. Show \( T \) is a linear transformation.
2. Determine if \( T \) is injective by computing the null space of the matrix representation \( M_T \).

T.3. [15 Points] Let \( V \) be a subspace of \( \mathbb{C}^n \) and define the orthogonal complement of \( V \) by \( V^\perp = \{ \vec{x} \in \mathbb{C}^n \mid \langle \vec{x}, \vec{v} \rangle = 0 \text{ for every } \vec{v} \in V \} \).

1. Show that \( V^\perp \) is a subspace of \( \mathbb{C}^n \).
2. Find a basis of \( V^\perp \) in the special case where \( V = \langle \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \rangle \subseteq \mathbb{C}^3 \).

T.4. [15 Points] Let \( T : M_{nn} \rightarrow M_{nn} \) be defined by \( T(A) = A^t \). Find all eigenvalues of \( T \) and describe all of the eigenspaces. [Hint: consider the linear transformation \( T \circ T \).]

T.5. [15 points] Professor Beezer has proven that if \( V \) is a finite-dimensional vector space and \( T : V \rightarrow V \) has \( \ker(T) = \{ \vec{0} \} \) then \( T \) is an isomorphism. Show that this is not necessarily the case if \( V \) is infinite dimensional by giving an example of a linear transformation \( T : P \rightarrow P \) that is not surjective but that has \( \ker(T) = \{ \vec{0} \} \). [Recall that \( P \) is the infinite dimensional vector space of all polynomials.]

You must do both of these problems ON THIS SHEET

R.1. [15 points] Prove that the set \( V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{C}^3 : -2x_1 + 4x_2 + 3x_3 = 0 \right\} \) is a subspace of \( \mathbb{C}^3 \) by applying the three-part test of Theorem TSS. Write your proof according to the standards of this semester’s writing exercises.

R.2. [15 points] Part of Theorem NPNT ("Nonsingular Products, Nonsingular Terms") says: If \( A \) and \( B \) are square matrices of the same size, and \( AB \) is nonsingular, then \( B \) is nonsingular. Construct a proof by contradiction of this fact and write your proof according to the standards of this semester’s writing exercises. (You may not use Theorem NPNT in your proof.)