Directions:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Exam 5

“Computational” Problems

C.1. Do one (1) of the following:

(a) Given the matrix

\[
A = \begin{bmatrix}
p & -q \\
q & p
\end{bmatrix}
\]

where \( p \) and \( q \) are real numbers with \( q \neq 0 \).

i. Show that the eigenvalues of \( A \) are \( \lambda_1 = p + iq \) and \( \lambda_2 = p - iq \).

ii. Determine a nonsingular matrix \( S \) and a diagonal matrix \( D \) for which \( S^{-1}AS = D \).

(b) Prove that \( T : P_2 \rightarrow P_2 \) defined by \((Tf)(x) = f(x + 1)\) is both a linear transformation and injective.

C.2. Find the matrix \( M_{B,B}^T \) of the linear transformation \( T : P_2 \rightarrow P_2 \) defined by \((Tf)(x) = f(x + 1)\) where \( B = \{1, x, x^2\} \) is the standard basis of \( P_2 \).

Do Two (2) of these “In text, class or homework” problems

M.1. Prove the third part (transitive property) of Theorem SER, Similarity is an Equivalence Relation:

Suppose \( A, B \) and \( C \) are square matrices of size \( n \). Then

(a) \( A \) is similar to \( A \). (Reflexive)

(b) If \( A \) is similar to \( B \), then \( B \) is similar to \( A \). (Symmetric)

(c) If \( A \) is similar to \( B \) and \( B \) is similar to \( C \), then \( A \) is similar to \( C \). (Transitive)

M.2. Prove Theorem EDELI, Eigenvectors with Distinct Eigenvalues are Linearly Independent:

Suppose that \( A \) is a square matrix and \( S = \{x_1, x_2, x_3, ..., x_p\} \) is a set of eigenvectors with eigenvalues \( \lambda_1, \lambda_2, \lambda_3, ..., \lambda_p \) such that \( \lambda_i \neq \lambda_j \) whenever \( i \neq j \). Then \( S \) is a linearly independent set.
M.3. Prove Theorem SSRLT, Spanning Set for Range of a Linear Transformation
Suppose that \( T : U \rightarrow V \) is a linear transformation and \( S = \{ u_1, u_2, u_3, ..., u_t \} \) spans \( U \). Then \( R = \{ T(u_1), T(u_2), T(u_3), ..., T(u_t) \} \) spans \( R(T) \).

M.4. Prove Theorem VRI, Vector Representation is Injective
If \( B = \{ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \} \) is a basis for the vector space \( V \) then The function \( \rho_B : V \rightarrow \mathbb{C}^n \) given in Definition VR [548] is an injective linear transformation.

Do One (1) of these “Not in Text” problems
T.1. Prove: If \( A \) is diagonalizable, then \( A^T \) is similar to \( A \).

T.2. This problem is Theorem CLTLT, Composition of Linear Transformations is a Linear Transformation in the textbook. Prove it, using the definition of linear transformation (you cannot just cite a theorem in the book.)

T.3. Define two vectors \( f, g \) in the vector space \( P_2 \) to be \textbf{orthogonal with respect to the coordinate basis} \( B = \{ 1, x, x^2 \} \) if \( \langle \rho_B(f), \rho_B(g) \rangle = 0 \). [Recall that \( \rho_B(f) \) is a vector in \( \mathbb{C}^2 \).] Find a basis for the set of all polynomials \( g \) in \( P_2 \) that are orthogonal with respect to the coordinate basis \( B \) to the polynomial \( f(x) = 2 + x \).

Cumulative Exam

Do Two (2) of these “In text, class or homework” problems
CC.1. (1 point each) If \( A \) is a square matrix, make a list of statements equivalent to “\( A \) is nonsingular”

CC.2. Let \( U, V \) be abstract vector spaces and \( T : U \rightarrow V \) a function. Show that \( T \) is a linear transformation \textbf{if and only if} for all \( \vec{u}_1, \vec{u}_2 \in U \) and all scalars \( a, b \) we have \( T(a\vec{u}_1 + b\vec{u}_2) = aT(\vec{u}_1) + bT(\vec{u}_2) \). [Be sure to prove both directions of the “if and only if”.

CC.3. Given an invertible matrix \( S \), prove the following transformation \( T : M_{nn} \rightarrow M_{nn} \) is linear.
\[
T(A) = S^{-1}AS
\]

CC.4. If there are square matrices \( A \) and \( B \) satisfying the property that \( B^2 = A \), then we say \( B \) is a \textbf{square root} of \( A \). It is easy to see that a diagonal matrix \( D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & d_{nn} \end{bmatrix} \) has \( \sqrt{d_{11}} \begin{bmatrix} 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & \sqrt{d_{nn}} \end{bmatrix} \) as a square root.

Prove that if \( A \) is a diagonalizable matrix, then \( A \) has a square root.

Do One (1) of these “Not in text” problems
MM.1. It is “obvious” that if \( a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_k \vec{v}_k = \vec{0} \) is a nontrivial relation of linear dependence and if \( a_i \neq 0 \), then \( \vec{v}_i \) is in the span of the remaining vectors. Use this fact to prove that if a set \( S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n \} \) is linearly dependent, then there is an index \( t \) for which \( \vec{v}_t \) is equal to a linear combination of the vectors \( \vec{v}_{t+1}, \vec{v}_{t+2}, \cdots, \vec{v}_n \) that \textbf{follow} it in \( S \).

MM.2. Use the principle of mathematical induction to prove the following fact we have used repeatedly throughout the semester.
Suppose \( V \) is a vector space, \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n \) and \( \vec{u}_1, \vec{u}_2, \vec{u}_3, \cdots, \vec{u}_n \) are vectors in \( V \). Then \( (\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n) + (\vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_n) = (\vec{v}_1 + \vec{u}_1) + (\vec{v}_2 + \vec{u}_2) + \cdots + (\vec{v}_n + \vec{u}_n) \) for every positive integer \( n \).
1. (10 points) Prove that the set $Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \bigg| 2x_1 - 4x_2 + x_3 = 0 \right\}$ is a subspace of $\mathbb{C}^3$ by applying the three-part test of Theorem TSS.

2. (10 points) Suppose that $A$ and $B$ are square matrices of the same size, and $AB$ is nonsingular. Give a proof by contradiction that $B$ is nonsingular. (Do not do this problem simply by quoting a theorem from the book.)