Technology used: ________________________________

Directions:
- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Exam 5

Do Both of these “Computational” Problems

C.1. (20 points) Is \( f(x) = 1 + x + x^2 + x^3 \) in the span of \( \{ 1 + 2x + 9x^2 + x^3, \ 9 + 7x + 7x^3, \ 1 + 8x + x^2 + 5x^3, \ 1 + 8x + 4x^2 + 8x^3 \} \)?

C.2. (10 points each) Given the linear transformation \( T : P_2 \to P_2 \) defined by \( T(p(x)) = p(x + 1) \).

(a) Find the matrix \( M_{B,B}^T \) where \( B = \{ 1, x, x^2 \} \)

(b) Find the algebraic and geometric multiplicities of all the eigenvalues of \( T \).

Do Two (2) of these “In text, class or homework” problems

M.1. (20 points) Prove Theorem VRS, Vector Representation is Surjective

If \( B = \{ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \} \) is a basis for the vector space \( V \) then The function \( \rho_B : V \to \mathbb{C}^n \) given in Definition VR is a surjective linear transformation.

M.2. (20 points) Suppose that \( V \) is a vector space and \( T : V \to V \) is a linear transformation. Prove that \( T \) is injective if and only if \( \lambda = 0 \) is not an eigenvalue of \( T \).

M.3. (20 points) Prove Theorem FTMR, Fundamental Theorem of Matrix Representation:

Suppose that \( T : U \to V \) is a linear transformation, \( B \) is a basis for \( U \), \( C \) is a basis for \( V \) and \( M_{B,C}^T \) is the matrix representation of \( T \) relative to \( B \) and \( C \). Then, for any \( \vec{u} \in U \), \( \rho_C(T(\vec{u})) = M_{B,C}^T(\rho_B(\vec{u})) \)

Do One (1) of these “Other” problems

T.1. (20 points) The Fibonacci sequence \( F_n \) is defined by the recursion \( F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \) for each \( n \geq 2 \). The first few terms of the sequence are \( 0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots \). It can be shown that the matrix \( A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \) has the property that \( [A^n]_{1,2} = F_n \). That is, for any nonnegative integer \( n \), the entry in the first row and second column of the \( n \)th power of \( A \) is the Fibonacci number \( F_n \). Show that \( A \) is diagonalizable and use the diagonal matrix to determine a closed form for \( F_n \).

[By closed form I mean a non-recursive formula.]
T.2. (20 points) Define two vectors $f, g$ in the vector space $P_1$ to be **orthogonal with respect to the coordinate basis** $B = \{1, x\}$ precisely when $(\rho_B(f), \rho_B(g)) = 0$. [Recall that $\rho_B(f)$ is a vector in $\mathbb{C}^2$.] Find a basis for the set of all polynomials $g$ in $P_1$ that are orthogonal with respect to the coordinate basis $B$ to the polynomial $f(x) = 2x$.

**Final Exam Cumulative**

**Do Two (2) of these “In text, class or homework” problems**

CC.1. Do **one** (1) of the following:

(a) (20 points) Prove that the vector spaces $M_{mn}$ and $M_{nm}$ are isomorphic. Use terminology and notation correctly.

(b) (20+ points) If $A$ is a square matrix, make a list of statements from Theorem NME, Nonsingular Matrix Equivalences. Points are taken off for incorrect statements. Extra credit for more than 10 correct statements.

(c) (20 points) Let $U, V$ be abstract vector spaces and $T : U \to V$ a function. Show that $T$ is a linear transformation **if and only if** for all $\vec{u}_1, \vec{u}_2 \in U$ and all scalars $a, b$ we have $T(a\vec{u}_1 + b\vec{u}_2) = aT(\vec{u}_1) + bT(\vec{u}_2)$.[Be sure to prove **both** directions of the “if and only if”.

CC.2. (20 points) Find a basis for the kernel of the linear transformation $T : P_2 \to \mathbb{R}^3$ given by

$$T(f) = \begin{bmatrix} f(0) \\ f'(1) \\ f(2) \end{bmatrix}.$$ 

CC.3. (20 points) The set $V = \text{span}\{\cos(t), \sin(t), t \cos(t), t \sin(t)\}$ is a basis for a subspace of the vector space of functions $F = \{f : \mathbb{C} \to \mathbb{C}\}$. Find the preimage of $\sin(t)$, $T^{-1}(\sin(t))$, under the linear transformation $T : V \to V$ given by $T(f) = f'$.

**Do Two (2) of these “Other” problems**

MM.1. (20 points) A linear transformation $T : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$ is given by $T(A) = \frac{1}{2}A + \frac{1}{2}A^\prime$. Find all of the distinct eigenvalues of $T$.

MM.2. (20 points) Suppose that $T : V \to V$ is a linear transformation. Prove that $(T \circ T)(\vec{v}) = \vec{0}$ for every $\vec{v} \in V$ if and only if $R(T) \subseteq K(T)$ (the range of $T$ is a subset of the kernel of $T$).

MM.3. (20 points) Recall that if $V$ is a subspace of $\mathbb{C}^n$, then the orthogonal complement of $V$ is the set $V^\perp = \{\vec{x} \in \mathbb{C}^n : \forall \vec{v} \in V, (\vec{v}, \vec{x}) = 0\}$. Show that $V^\perp$ is a subspace of $\mathbb{C}^n$.

MM.4. (20 points) Recall that if $V$ is a subspace of $\mathbb{C}^n$, then the orthogonal complement of $V$ is the set $V^\perp = \{\vec{x} \in \mathbb{C}^n : \forall \vec{v} \in V, (\vec{v}, \vec{x}) = 0\}$. Let $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for a subspace $V$ of $\mathbb{C}^n$. Show that if $\vec{x} \in \mathbb{C}^n$ satisfies $(\vec{v}_i, \vec{x}) = 0$, for all of the basis vectors $\vec{v}_i$, $i = 1, \ldots, p$ then $\vec{x} \in V^\perp$. That is, $\vec{x}$ is perpendicular to **every** vector in $V$ and not just the vectors in the basis $B$. 

2
You MUST do both of these problems.

Show your work on this page.

1. (10 points) Prove that the set \( Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left| \begin{array}{c} 2x_1 - 4x_2 + x_3 = 0 \end{array} \right. \right\} \) is a subspace of \( \mathbb{C}^3 \) by applying the three-part test of Theorem TSS.

2. (10 points) Suppose that \( A \) and \( B \) are square matrices of the same size, and \( AB \) is nonsingular. Give a proof by contradiction that \( B \) is nonsingular. (Do not do this problem simply by quoting a theorem from the book.)