Name

Technology used:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.
- When given a choice, be sure to specify which problem(s) you want graded.

Do any three (3) of these computational problems

C.1. Do all of the following.

(a) Show that the set of vectors \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\} \) is linearly dependent.

(b) Find two vectors \( \vec{w}_1, \vec{w}_2 \) that are both in \( S \) and for which \( < S > = < T > \), where \( T = \{ \vec{w}_1, \vec{w}_2 \} \).

(c) Write one of the extra vectors in \( S \) as a linear combination of \( \vec{w}_1, \vec{w}_2 \).

C.2. Write all of the following complex numbers in the form \( a + bi \).

(a) \( 2 (2 - 3i) - 7 (6 + 2i) \)

(b) \( \frac{4 + 3i}{2 - i} \)

(c) \( \sqrt{i} \) [Hint: write \( (a + bi)^2 = i \) and solve a system of equations.]

C.3. The vectors \( \vec{u}_1, \vec{u}_2, \vec{u}_3 \) below form an orthonormal set. Use the Gram-Schmidt procedure to find a vector \( \vec{u}_4 \) so that \( \{ \vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4 \} \) is an orthonormal set which has the same span as \( \{ \vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{v}_4 \} \).

\[
\begin{align*}
\vec{u}_1 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \\
\vec{u}_2 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \\
\vec{u}_3 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \\
\vec{v}_4 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

The Gram-Schmidt formula is

\[
\vec{u}_i = \vec{v}_i - \left( \frac{\langle \vec{v}_i, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \right) \vec{u}_1 - \cdots - \left( \frac{\langle \vec{v}_i, \vec{u}_{i-1} \rangle}{\langle \vec{u}_{i-1}, \vec{u}_{i-1} \rangle} \right) \vec{u}_{i-1}
\]

C.4. Compute the following matrix-vector product by hand in two ways.

\[
\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.
\]
Do any two (2) of these problems from the text, homework, or class. You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Suppose \( S = \{u_1, u_2, \ldots, u_p\} \) is a linearly independent set and that \( v \notin \langle S \rangle \). Prove the set \( W = \{u_1, u_2, \ldots, u_p, v\} \) is a linearly independent set.

M.2. Suppose \( S = \{v_1, v_2, v_3\} \) is a linearly independent set in \( R^5 \). Is the set of vectors \( 2v_1 + v_2 + 3v_3, v_2 + 5v_3, 3v_1 + v_2 + 2v_3 \) linearly dependent or independent?


Suppose that \( A \) and \( B \) are \( m \times n \) matrices. Then \( (A + B)^t = A^t + B^t \).

Do one (1) of these problems you’ve not seen before.

T.1. Suppose \( A \) is a square matrix of size \( n \) satisfying \( A^2 = AA = O \). Prove that the only vector \( \vec{x} \) satisfying \( (I_n - A) \vec{x} = \vec{0} \) is the zero vector.

T.2. Recall that \[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
Now explain why the fact that \[
\begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -4 & -2 & -2 & 0 & 1 & 0 \\ -5 & -2 & -4 & 0 & 0 & 1 \end{bmatrix}
\]
has reduced row-echelon form \[
\begin{bmatrix} 1 & 0 & 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 0 & -\frac{5}{2} & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}
\]
tells us the only vectors \[
\begin{bmatrix} a \\ b \\ c \end{bmatrix}
\]
that can be in the span of \( S = \left\{ \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix} \right\} \) are those where \( a + 2b - c = 0 \).