Do any ten (10) of the following.

1. [15 points] Write out one Riemann Sum with \( n = 5 \) and \( \|P\| = 2 \) for the function \( f(x) = x^2 - x \) on the interval \([0, 6]\). You do not need to simplify any of the algebra.

2. [10, 5 points] Here is a Riemann sum for a function \( f \) on an interval \([a, b]\): \( R_n = \frac{1^7 + 2^7 + 3^7 + \cdots + n^7}{n^8} \)
   
   (a) What are \( f \) and \([a, b]\)?
   
   (b) Use the definition of the definite integral to evaluate the limit \( \lim_{n \to \infty} \frac{1^7 + 2^7 + 3^7 + \cdots + n^7}{n^8} \).
3. [15 points] The region in the first quadrant enclosed by the coordinate axes, the curve \( y = \ln(x) \), and the line \( y = 1 \) is revolved around the \( y \)-axis to generate a solid. Find the volume of the solid.

4. [15 points] Find the length of the enclosed loop \( x = t^2, \ y = (t^3/3) - t \) where \( t \) starts at \(-\sqrt{3}\) and ends at 0.
5. [15 points] Evaluate the following integral

\[ \int \frac{x^4 + 3x^3 + 4x^2 + 13x + 3}{x^3 + 4x} \, dx \]

6. [15 points] The population of the world was estimated to be 3 billion in 1959 and 6 billion in 1999. What would an exponential model of population growth predict the population to be in 2008? [The actual population was 6,747,510,603 as of 17:44 GMT (EST+5) Dec 16, 2008 – data from the US Census Bureau’s population clock.]
7. Recall that \( \int_a^b u \, dv \bigg|_a^b = \int_a^b u \, dv - \int_a^b v \, du \). Use this and the fact that \( \int_0^\infty t^2e^{-t} \, dt = 2 \) to evaluate the improper integral \( \int_0^\infty t^3e^{-t} \, dt \)

(a) [5 points] Antiderivative
(b) [10 points] Limit(s)

8. [5, 5, 5 points] Give examples of the following.

(a) A diverging infinite sequence.
(b) A converging infinite series.
(c) An improper integral with at least 3 improprieties.
9. [15 points] Determine the radius of convergence, center, values of $x$ where the series converges absolutely, and values of $x$ where the series converges conditionally for $\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{4^n \sqrt{n+1}}$.

10. [15 points] Find the Taylor series generated by the function $f(x) = \cos(x)$ at $x = a$ where $a = \pi/2$. Be careful. Use the full process for finding a Taylor series (taking derivatives of all orders). **Do not** use the Maclaurin series for $\cos(x)$ since that is not centered at $a = \pi/2$.!
11. [15 points] Starting with the Maclaurin series for \( \sin(x) \) in the “Useful Information” section below, approximate the value of the integral \( \int_0^1 t \sin(t^4) \, dt \). Use an error bound (not your calculator) to prove your approximation has an error of magnitude less than \( 10^{-6} \).

12. [15 points] Prove that \( \lim_{n \to \infty} \frac{x^n}{n!} = 0 \) for every number \( x \). [Hint: the Maclaurin series for \( e^x \) converges for every number \( x \).]

Useful Information

- The Taylor Series for \( f(x) \) at \( x = a \) is \( \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a) (x - a)^n \)
- The error bound for a converging alternating series \( \sum_{k=1}^{\infty} (-1)^{k+1} u_k \) is \( |L - S_n| < u_{n+1} \)
- \( \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \) converges to \( \sin(x) \) for every real number \( x \).
- \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) for every real number \( x \).