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Geometry 300
Term Paper
April 17, 2002

The Pythagorean Theorem and Euclid's Fifth Postulate

History:

The Pythagorean theorem reads, "The square described upon the hypotenuse of a right-angled triangle is equal to the sum of the squares described upon the other two sides."[1] The ideas behind this theorem, which has been attributed to Pythagoras of Samos, who lived during the sixth century B.C., were being used long before Pythagoras' existence. There is evidence on Babylonian clay tablets to indicate that the results of the Pythagorean theorem were being used as early as the sixteenth century B.C.[2] Pythagoras, however, was the first attributed to the geometrical construction of the Pythagorean theorem.[3] "Pythagoras was regarded by his contemporaries as a religious prophet."[4] He started a cult, which ultimately believed that by studying music and mathematics one could be closer to God. Pythagoras also started a school, from which much of much of his work has been extracted. The Pythagorean school gave Euclid the systematic foundation of plane geometry and lasted until 400 B.C.[5]

It wasn't until around 300 B.C. that Euclid produced the *Elements*. In producing the first four books of the *Elements* Euclid used many ideas and results given by the Pythagorean school. Although Pythagoras came long before Euclid, and the

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[1] History
[2] History
[3] History
[4] Greenburg, p.7
[5] Greenburg, p.8

Pythagorean theorem long before Euclid's fifth postulate, it was never deduce, during the time of the Pythagorean school, that the Pythagorean theorem only held in Euclidean physical space.[6] Therefore—concluding that the Pythagorean theorem only holds if Euclid's fifth postulate also holds.

Euclid's fifth postulate comes from Euclid's first book of the *Elements*,[7] and reads,

> "If two line are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180 degrees, then the two lines meet on that side of the transversal."[8]

However, unlike the other four postulates stated in the *Elements*, many historians felt the truth of Euclid's fifth postulate to be unobvious.[9] Book I of the *Elements* is set up so that Euclid's fifth postulate is not invoked until it is absolutely necessary (although if used from the beginning it would have simplified the proofs of many other theorems). Then once invoked every theorem following, with the exception of one (it is possible to construct parallel lines), depends on Euclid's fifth postulate.[10] This construction of Book I led many historians to question Euclid's own confidence in assuming the fifth postulate rather than deducing it from the others. Many historians attempted to deduce Euclid's fifth postulate since it's existence as an axiom was so controversial. In doing so, historians have proven that Euclid's fifth postulate is equivalent to the Pythagorean theorem among others.

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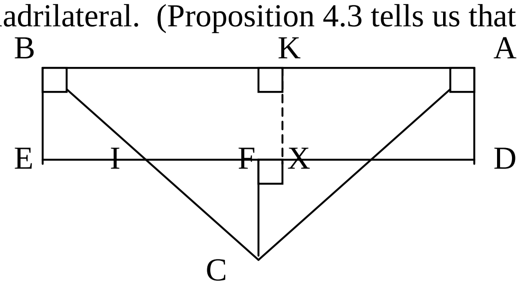
[6] Adler, p.254
[7] Martin, p.123
[8] Greenburg, p.128
[9] Trudeau, p.118
[10] Trudeau, p.85, 118-119

Application:

Here we will deduce that the Pythagorean theorem is equivalent to Euclid's fifth postulate. Let us first denote that it is proven in exercises 18-21, in chapter five of Greenburg, that Hilbert's parallel postulate implies the Pythagorean theorem.[11] We also know that Euclid's fifth postulate is equivalent to Hilbert's parallel postulate.[12] (Also, note that Hilbert's parallel postulate does not hold in Hyperbolic geometry.) Therefore, we will now prove that the Pythagorean theorem only holds in Euclidean geometry, and therefore is equivalent to both Hilbert' parallel postulate and Euclid's fifth postulate.

The steps to prove that the Pythagorean theorem only holds in Euclidean geometry are as follows:

(a) Given triangle ΔABC, let I,J, and K be the midpoints of BC, CA, and AB, respectively. Drop perpendiculars AD, BE, and CF from the vertices to line IJ. Prove that AD ≅ CF ≅ BE, and, hence that quadrilateral EDAB is a Saccheri quadrilateral. (Proposition 4.3 tells us that I,J and K are unique.)



Given ΔCFJ and ΔADJ we know that they are right triangles, since CF and AD perpendicular to line IJ. We know that AJ≅CJ since J is the midpoint of AC. We also know, by proposition 3.15 in Greenburg, that angles <FJC and <DJA are

[11] Greenburg, p.170-171
[12] Greenburg, p.128

congruent. Therefore, by proposition 4.1 in Greenburg (SAA), we know that $\Delta CFJ \cong \Delta ADJ$.

Given ΔCFI and ΔBEI we know that they are right triangles, since CF and BE perpendicular to line IJ. We know that $BI \cong CI$ since I is the midpoint of BC. We also know, by proposition 3.15 in Greenburg, that angles <BIE and <CIF are congruent. Therefore, by proposition 4.1 (SAA), we know that $\Delta CFI \cong \Delta BEI$. We also know, by corresponding parts of congruent triangles, that $CF \cong AD$ and $CF \cong BE$, hence, $AD \cong BE$.

(b) Prove that the perpendicular bisector of AB (i.e., the perpendicular through K) is also perpendicular to line IJ, and, hence, that line IJ is divergently parallel to line AB.

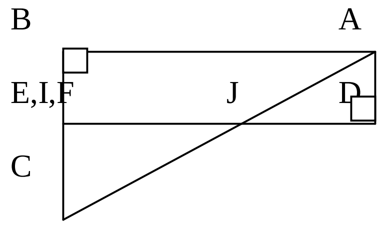
The perpendicular bisector of AB, call it m, hits line IJ at a unique point, call it X (proposition 2.1). By incidence axiom one we know that lines BX and AX exist. This creates congruent, right triangles ΔKXA and ΔKXB, by SAS. ($AK \cong BK$, $<AKX \cong BKX$, $KX \cong KX$) Therefore, by corresponding parts of congruent triangles we know that $BX \cong AX$ and $<KBX \cong KAX$. Then, by proposition 4.2 in Greenburg, we also know that triangles ΔADX and ΔBEX are congruent. This gives us that angles <EBX and <DAX are congruent. Hence, by angle addition, we know that angles <EBK and <DAK are congruent.

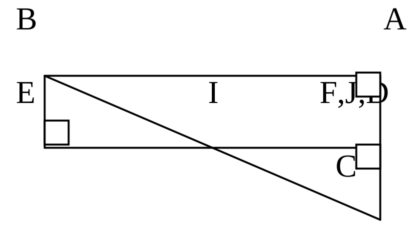
Since AD≅BE and angles <EBK and <DAK are congruent, we know that triangles ΔKBE and ΔKAD are congruent, by SAS. Therefore, <AKD≅<BKE and KD≅KE by corresponding parts of congruent triangles. Since m is perpendicular to line AB we know, by angle subtraction, that angles <DKX and <EKX are congruent. Therefore, by SAS, we know that triangles ΔDKX and ΔEKX are also congruent. This gives us that angles <DXK and <EXK must be congruent, by corresponding parts of congruent triangles. However, angles <DXK and <EXK are supplementary angles, and by definition of a right angle we know that angles <DXK and <EXK must be right angles. Therefore, line AB is parallel to line IJ.

(c) Prove that the length of segment IJ = ½ the length of ED. Deduce that in hyperbolic geometry that the length of segment IJ is strictly less than ½ the length of segment AB.

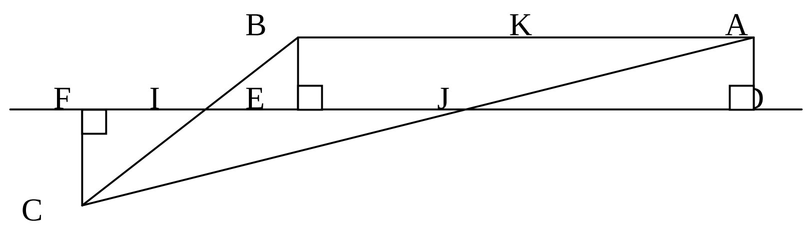
Here we must consider the three cases where I*F*J, F=I or J, and where F*I*J or I*J*F.

For the case in which F is between I and J we can proceed as follows:
Given ΔADJ ≅ ΔCFJ and ΔBEI≅ΔCFI, by previous steps, we know that FJ≅JD and IF≅EI. Therefore, by segment addition we know that the length of IF plus the length of FJ is equal to the length of JD plus the length of EI. Therefore, we can conclude that the length of ED = ½ the length of IJ.





If F=I or J then E=I=F or F=J=D. Then we know that triangles $\Delta CJF \cong \Delta AJD$ or $\Delta CIF \cong \Delta BEI$. By corresponding parts of congruent triangles we know that $IJ \cong JD$, for F=I, and $EI \cong IJ$ for F=J. Therefore we know that, for F=I, the length of IJ is one half the length of ID=ED. For F=J we know that the length of IJ is one half the length of EJ=ED.[13]



If angles <A or <B is obtuse the proof is as follows:

Rename if necessary to allow <B to be the obtuse angle. We know that angle <C is congruent to itself. We have already found that triangles ΔCFI and ΔBEI, along with, ΔAJD and ΔCFJ are congruent. Therefore, we know that $FI \cong IE$, $FJ \cong JD$. Using segment subtraction and segment addition we can use the following algebraic expressions to deduce IJ= ½ ED.

$$FI=IE$$
$$FJ=JD$$
$$FD=2JD=2FJ$$
$$FJ=FI+IE+EJ=2IE+EJ$$
$$JD=2FI+EJ=2IE+EJ$$
$$IJ=IE+EJ$$
$$ED=JD+EJ$$
$$ED=EJ+2IE+EJ$$
$$ED=2EJ+2IE=2(EJ+IE)=2IJ$$

[13] Trudeau, p.136

We know that lines AB and IJ are parallel, therefore, quadrilaterals KBEX and KADX are Lambert. By the second corollary to theorem 4.4 in Greenburg we know that angles <KBE and <KAD are <= 90 degrees. However, if either angle <KBE or <KAD = 90 degrees, either quadrilateral KBEX or KADX is a rectangle, and by theorem 6.1 in Greenburg we know that there are no rectangles in hyperbolic geometry. Therefore, angles <KBE and <KAD must be less than 90 degrees.

Since angles <KBE and <KAD are less than 90 degrees, we know by exercise 2 of chapter 5 in Greenburg applied to quadrilaterals KADX and KBEX that segments AK<DX and KB<XE. Therefore, by segment addition we know that AB<ED. Earlier we proved that the length of IJ = ½ the length of ED. Therefore, the length of IJ < ½ the length of AB.

(d) Supposing that angle <C is a right angle we can prove that the Pythagorean theorem does not hold in hyperbolic geometry.

Given one angle of ΔABC is 90 degrees, we know, by theorem 4.4 in Greenburg, that the other two angles must be < 90 degrees.

To prove that the Pythagorean theorem does not hold in hyperbolic geometry, let's assume that it does hold (RAA hypothesis).[14] Applying the Pythagorean theorem to triangles ΔCAB and ΔCJI we get:

$$AB^2 = CA^2 + CB^2$$
$$IJ^2 = CI^2 + CJ^2$$

Since CI = ½ CB and CJ = ½ CA we get:

$$IJ^2 = (\tfrac{1}{2} CB)^2 + (\tfrac{1}{2} CA)^2$$
$$IJ^2 = \tfrac{1}{4} CB^2 + \tfrac{1}{4} CA^2$$
$$IJ^2 = \tfrac{1}{4} (CB^2 + CA^2) = \tfrac{1}{4} AB^2$$
$$IJ^2 = \tfrac{1}{4} AB^2$$
$$IJ = \tfrac{1}{2} AB$$

However, if IJ = ½ AB then the length of AB = ED, which we proved (previously) cannot be possible in hyperbolic geometry.

(e) Suppose instead that AC≅BC. Then we can prove that K,F and C are collinear but that F is not the midpoint of CK. (This makes ΔCAB isosceles, and therefore all angles of ΔCAB are acute. Therefore, we know that F is between I and J.)

We know that $\Delta BEI \cong \Delta CFI$ and $\Delta CFJ \cong \Delta ADJ$. If AC≅BC we know that BI≅IC≅CJ≅JA, by definition of midpoints. Therefore, $\Delta CFI \cong \Delta CFJ$ by proposition 4.2 in Greenburg. This then tells us that $\Delta CFI \cong \Delta CFJ \cong \Delta BEI \cong \Delta ADJ$. Before we found that IJ = ½ED. By corresponding parts of congruent triangles we know that EI ≅ JD ≅ FJ ≅ FI. By segment addition, and I*F*J, we know that IJ = 2FJ. Since, IJ = ½ED, ED = 4FJ. Therefore, by definition of midpoint we know that F is the midpoint of ED, which makes triangles ΔBEF and ΔADF congruent, by SAS.

[14] Trudeau p.220-221

Now we know that X and K are collinear as well as X and F. We also found in part (b) that triangles ∆KXA and ∆KXB are congruent. By corresponding parts of congruent triangles we know that AX ≅ BX. Since, angles <BEI and <ADJ are both right angles, and BE≅AD (by previous results), triangles ∆BXE and ∆AXD must be congruent. Therefore, by corresponding parts of congruent sides EF≅FD≅XD≅EX. Since ∆CAB is isosceles by construction we know E*F*D and E*X *D. Hence, X=F and C, F and K are all collinear.

Now line KF is the perpendicular bisector of segment AB, and line CF is perpendicular to line ED. In part (a) we found that AD≅BE and in part (b) that lines AB and ED are parallel, thus, by lemma 6.2 in Greenburg, KF < AD≅BE. Since, CF≅AD≅BE, we know that F is not the midpoint of CK.

<u>Extensions</u>:

This proof that the Pythagorean theorem is equivalent to Euclid's fifth postulate is also used in practical applications of forces. "[The] usual rule for adding two equal forces acting at the ends of a line segment is equivalent to Euclid's fifth postulate."[15] In mechanics the line segment indicated would correspond to segment ED in Figure 1 and the two forces, rays DA and EF, in the upward direction.

[15] Adler, p.253

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