Sets and Logic

Set Basics

- Notation for standard sets of numbers: \( \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N} \)
- Standard operators on sets: \( \in, \cup, \cap, \subseteq, \notin \)
- “Set Builder” notation: \{Universal set | defining restriction\}
  1. Example: The set of even integers: \( \{n \in \mathbb{Z} | n = 2k \text{ and } k \in \mathbb{Z}\} \)
  2. Example: The set of Real-valued functions whose domain is the set of Real numbers and whose graph passes through the point \((2, 5)\): \( \{f : \mathbb{R} \to \mathbb{R} | f(2) = 5\} \).

Logical Operators and their Truth Tables

1. Not: (negation): \(\neg\)
2. And (Conjunction): \(\land\)
3. Or: (Disjunction): \(\lor\)
4. Conditional (Implication): \(\implies\)
5. Equivalence (If and only if): \(\iff\) (\(\equiv\))

Tautologies / Contradictions

1. \(p \land \neg p\)
2. \(p \lor \neg p\)
3. \(\neg\neg p \iff p\)
4. \((P \land (P \implies Q)) \implies Q\)
5. \((p \lor q) \iff (\neg p) \land (\neg q)\)
6. \((p \land q) \iff (\neg p) \lor (\neg q)\)
7. \((p \implies q) \iff (\neg q) \implies (\neg p)\) contrapositive
8. \((p \implies q) \iff (\neg q) \lor q\)
9. \(((P \land \neg Q) \implies (R \land \neg R)) \iff (P \implies Q)\)
10. \(((p \implies q) \implies (r \implies s)) \iff ((p \implies q) \land r) \implies s\)

Quantifiers

Universal: \(\forall\) Example: \(\forall x \in \mathbb{R} \quad x^2 + 1 > 0\) is a true statement

Existential: \(\exists\) Example: There is an integer solution to \(x^2 + 5x + 6 = 0\) is a true statement. \((x = -2)\)

Negation of quantifiers \(\neg\exists x \ (p(x))\) means \(\forall x \ \neg p(x)\)

Proof Methods

Direct Proof of \(H \implies C\) or \(H \implies C_1 \land C_2\)

1. Start with the (conjoined) hypotheses of \(H\)
2. Use nothing but logical steps See below.
3. Deduce \(C\). (Deduce each of the \(C_i\))
Use of the Contrapositive to prove $H \implies C$ Uses the tautology $(H \implies C) \iff (\sim C) \implies (\sim H)$

1. Start with (conjoined) statements of $\sim C$
2. Use nothing but logical steps
3. Deduce $\sim H$

Proof by Contradiction of $H \implies C$ Uses Tautology \[((H \land (\sim C))) \implies (D \land (\sim D))) \implies C\]

1. Start with $(\sim C)$
2. Use $H$ and nothing but logical steps to get $(D \land (\sim D))$
3. Deduce $(\sim C)$

How to deal with conjunctions and disjunctions

Disjoined Hypotheses $H_1 \lor H_2 \implies C$ Uses the Tautology ... 

1. Do it by cases: Prove the 2 individual implications $H_i \implies C$

Disjoined Conclusions $H \implies C_1 \lor C_2$ Uses the Tautology \[H \implies (C_1 \lor C_2) \iff [(H \land \sim C_1) \implies C_2]\]

1. Start with $C$ and the negation of all but one $C_i$
2. Deduce the last $C_i$

How to prove Universal statements $\forall x \ (p(x) \implies q(x))$

1. Start with an arbitrary element $x$ in the universal set $X$
2. Show that, using only the properties of being in $X$ $p(x) \implies q(x)$
3. Example: If $x > 1$ then $x^2 > x$.
   Proof: Let $x$ be an arbitrary number bigger than 1.

How to prove Existential Statements $\exists x \ p(x)$

1. Best approach is to actually exhibit an instance of $x$.
2. Or do a proof by contradiction.

Forward-Backward method for doing proofs

Basic (Named) Rules of Inference

1. Modus Ponens (mode that affirms) (mode that affirms by affirming) \[ (p \implies q) \land p \implies q \]
2. Syllogism \[ ((p \implies q) \land (q \implies r)) \implies (p \implies r) \]
3. Contrapositive \[ (p \implies q) \iff ((\sim q) \implies (\sim p)) \]
   (a) converse, obverse

4. Modus Tollens (mode that denies) ("the way that denies by denying") \[ (p \implies q) \land (\sim q) \implies (\sim p) \]
5. Contradiction \[ (((p \land (\sim q))) \implies (r \land (\sim r))) \implies q \]