1. [10, 10, 10 points] Evaluate the following integrals. Show all of your work.

1. \( \int \cos^5(3x) \, dx = \int [\cos^2(3x)]^2 \cos(3x) \, dx = \frac{1}{3} \int (1 - \sin^2(3x))^2 \, d(\sin 3x) = \frac{1}{3} \int (1 - 2u^2 + u^4) \, du = \frac{1}{3}u - \frac{2}{3}u^3 + \frac{1}{15}u^5 + C. \) Now backsubstitute \( u = \sin(3x). \)

2. \( \int \sec^4(2x) \, dx = \int [\sec^2(2x)] \sec^2(2x) \, dx = \frac{1}{2} \int (1 + \tan^2(2x)) \, d(\tan 2x) = \frac{1}{2} \int (1 + u^2) \, du = \frac{1}{6}u^3 + \frac{1}{2}u + C. \) Now backsubstitute \( u = \tan(2x). \)

3. \( \int y \ln(y) \, dy = \frac{1}{2}y^2 \ln(y) - \frac{1}{2}y^2 + C \)

(a) Where we used integration by parts and \( u = \ln(y), \, dv = y, \, du = \frac{1}{y}dy, \, v = \frac{1}{2}y^2 \)

2. [15 points] Find the length of the curve \( y = x^{1/2} - (1/3)x^{3/2}, \, 1 \leq x \leq 4. \)

1. Set \( x = t \) and \( y = t^{1/2} - (1/3)t^{3/2}, \) then \( \left[ \frac{dx}{dt} \right]^2 = [1]^2 = 1 \) and \( \left[ \frac{dy}{dt} \right]^2 = \left[ \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \right]^2 = \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x. \)

2. So

\[
ds = \sqrt{\left[ \frac{dx}{dt} \right]^2 + \left[ \frac{dy}{dt} \right]^2} \, dt = \sqrt{1 + \left( \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x \right) dt}
= \sqrt{\frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x} \, dt = \sqrt{\left( \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2} \right)^2} \, dt = \left| \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2} \right| \, dt
\]

3. So \( s = \int_1^4 \left| \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2} \right| \, dt = \int \left( \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2} \right) \, dt = x^{1/2} + \frac{3}{8}x^{3/2} \bigg|_1^4 = \frac{10}{3} \)

3. [15 points] Find the area of the surface generated by revolving the curve \( y = \sqrt{4x-x^2}, \, 1 \leq x \leq 2 \)

about the \( x \)-axis.

1. Set \( x = t \) and \( y = (4t - t^2)^{1/2} \) so that

\[
\left[ \frac{dx}{dt} \right]^2 + \left[ \frac{dy}{dt} \right]^2 = 1 + \left[ \frac{1}{2} \frac{(4 - 2t)}{\sqrt{4t - t^2}} \right]^2 = 1 + \frac{(2-t)^2}{4t - t^2} = \frac{4t - t^2 + (2-t)^2}{4t - t^2} = \frac{4}{4t - t^2}
\]

2. So, the surface area is \( 2\pi \int_1^2 (radius) \, ds = 2\pi \int_1^2 \sqrt{4t - t^2} \sqrt{\frac{4}{4t - t^2}} \, dt = 2\pi \int_1^2 \frac{2}{d} \, dt = 4\pi \)

4. [15 points] Solve the initial value problem \( \frac{dy}{dx} = \frac{y \ln(y)}{1 + x^2}, \, y(0) = e^2. \)
1. Separate variables to obtain \[ \int \frac{1}{y \ln(y)} \frac{dy}{dx} \] \[ = \int \frac{1}{1 + x^2} \ dx \] and use the substitution \( u = \ln(y) \), \( du = \frac{1}{y} dy \) on the left integral.

2. \( \int \frac{1}{u} \ du = \ln |u| + C_1 = \ln |\ln y| + C_1 = \arctan(x) + C_2 \). Setting \( C = C_2 - C_1 \) we get

3. \( \ln |\ln y| = \arctan(x) + C \) and the initial condition tells us that \( \ln |\ln (e^2)| = \arctan(0) + C \) so \( C = \ln (\ln (e^2)) = \ln(2) \).

4. So \( \ln |\ln y| = \arctan(x) + \ln(2) \) which implies

\[ \ln y = e^{\arctan(x)+\ln(2)} = e^{\arctan(x)} \cdot e^{\ln(2)} \]

So, \( y = e^{2\arctan(x)} \).

6. [10 points each] A deep dish-apple pie, whose internal temperature was 220°F when removed from the oven was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie’s internal temperature was 180°F. How much longer did it take for the pie to cool to 70°F?

1. Using \( T(t) - A = (T_0 - A) e^{-kt} \) with \( A = 40 \) and \( T_0 = 220 \) and \( T(15) = 180 \) we get

\[ 180 - 40 = (220 - 40) e^{-k(15)} \]

\[ \ln \left( \frac{7}{8} \right) = k \]

2. Then using this \( k \) and solving for \( t \) in

\[ 70 - 40 = (220 - 40) e^{-kt} \]

\[ \ln \left( \frac{1}{6} \right) = -kt \]

\[ t = -\ln \left( \frac{1}{6} \right) / \ln \left( \frac{7}{8} \right) \approx 106.9 \text{ minutes} \]

3. The answer is 106.9 - 15 = 91.9 minutes.

7. [15 points] A disk of radius 2 is revolved around the \( y \)-axis to form a solid sphere. A round hole of radius \( \sqrt{3} \), centered on the \( y \)-axis is bored through the sphere. Find the volume of material removed from the sphere.

1. Using cylindrical shells we see the volume removed from the sphere is \( 2\pi \int_0^{\sqrt{3}} x \pi - x^2 \ dx \) which we can integrate using \( u = 4 - x^2, du = -2x \ dx \). The removed volume is \( 2\pi \int_0^{\sqrt{3}} x\pi - x^2 \ dx = \frac{14}{3} \pi \)

Extra Credit [5 points] At each point on the curve \( y = 2\sqrt{x} \), a line segment of length \( h = y \) is drawn perpendicular to the \( xy \)-plane. Set up an integral that equals the area of the surface formed by these perpendiculars from \( x = 0 \) to \( x = 3 \). [Note that this is not a surface of revolution so none of the formulas in Chapter 6 apply. Develop your own integral by using Riemann sums to estimate the area of the surface.]

1. The surface extends vertically upward from the curve \( y = 2\sqrt{x} \). If we partition the graph of \( y = 2\sqrt{x} \) into many small arcs of length approximately \( \Delta s_k \), then the area of the surface above the \( k \)th arc is approximately \( 2\sqrt{x_k} \Delta s_k \). Thus the associated Riemann sum that approximates
the total area is \( \sum_{k=1}^{n} 2\sqrt{x_k} \Delta s_k \) and since \( f(x) = 2\sqrt{x} \) is a smooth curve on the given domain we know that the limit of Riemann sums exists and is equal to the integral \( \int_{0}^{3} 2\sqrt{x} \, ds \). To compute this actual area, we need to compute \( ds = \sqrt{1 + \frac{1}{x}} \, dx = \frac{x^{1/2}}{\sqrt{x+1}} \), so the integral is

\[
\int_{0}^{3} 2x^{1/2} \cdot \frac{x^{1/2}}{(x + 1)^{1/2}} \, dx = 2 \int_{0}^{3} \frac{x}{(x + 1)^{1/2}} \, dx
\]

which, when integrated by using the "Rule of Thumb" substitution \( u = x + 1 \), yields a value of \( \frac{16}{\pi} \).