Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

“Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them.”
— *Henri Poincaré*

### Problems

1. **You Must Do This Problem** Let $\tau = (a_1, a_2, \cdots, a_k)$ be a cycle of length $k$.
   (a) Prove that if $\sigma$ is any permutation, then $\sigma \tau \sigma^{-1} = (\sigma(a_1), \sigma(a_2), \cdots, \sigma(a_k))$.
   (b) Let $\mu$ be a cycle of length $k$. Prove there is a permutation $\sigma$ such that $\sigma \tau \sigma^{-1} = \mu$.

2. Do all three of the following. Prove that any element in $S_n$ (where $n \geq 3$) can be written as a finite product of
   (a) the transpositions $(12), (13), \cdots, (1n)$.
   (b) the transpositions $(12), (23), \cdots, (n-1, n)$
   (c) the two distinct cycles $(12), (12 \cdots n)$

3. Let $\sigma \in S_X$. For any $x, y \in X$, we say $x \sim y$ if there is an integer $n$ such that $\sigma^n(x) = y$.
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Let $x \in X$ and $\sigma \in S_X$ and define the **orbit of $x$ under $\sigma$** to be the set $O_{x, \sigma} = \{\sigma^n(x) : n \in \mathbb{Z}^+\}$. Prove that $O_{x, \sigma}$ is the equivalence class of $x$ under the equivalence relation $\sim$.
   [Note that this gives us a way to use a group to partition a set. We will see much more about this later.]
   (c) Let $X = 1, 2, 3, 4, 5, 6$ be the set of faces of a cube where, as viewed from a fixed location, $1, 2, 3, 4, 5, 6$ denote the front, right, back, left, top, and bottom faces respectively. Compute the orbit of face 1 under the element $\alpha = (13)(24) \in S_4$
   (d) Using the same set $X$ as above, find all of the equivalence classes $O_{x, \sigma}$ in $X$ where $\sigma = (124) \in S_4$.  

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