VS-2 (Section PD)
Suppose \( S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_{p-1}, \vec{v}_p, \vec{v}_{p+1} \cdots \vec{v}_m \} \) is an orthonormal basis for \( \mathbb{C}^m \) and let \( V = \langle \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_{p-1}, \vec{v}_p \} \rangle \) be the subspace of \( \mathbb{C}^m \) spanned by the first \( p \) vectors in \( S \) and \( W = \langle \{ \vec{v}_{p+1} \cdots \vec{v}_m \} \rangle \) be the subspace of \( \mathbb{C}^m \) spanned by the last \( m - p \) vectors in \( S \).

1. Quote the theorem from our textbook that tells us that \( \mathbb{C}^m = V \oplus W \).
2. Prove that if \( \vec{w} \in W \), then \( \vec{w} \) is orthogonal to every vector in \( V \).
3. Prove that if \( \vec{x} \) is orthogonal to every vector in \( V \), then \( \vec{x} \in W \).

Notes

- Because \( W \) satisfies the two properties (2 and 3) above, it is called the **orthogonal complement** of \( V \) in \( \mathbb{C}^m \) and is usually written \( V^\perp \).

- Professor Beezer has proved that

  1. Every subspace, \( V \), of \( \mathbb{C}^m \) has a basis  
  2. That basis can be extended to a basis of \( \mathbb{C}^m \) and  
  3. The Gram-Schmidt procedure can transform any basis into an orthonormal basis.

Your work along with these details proves the theorem

**Theorem 1** If \( V \) is a subspace of \( \mathbb{C}^m \) then \( \mathbb{C}^m = V \oplus V^\perp \)