Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. Only write on one side of each page.

“Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them.”
— Henri Poincaré

Problems

1. You must do this problem.
   
   (a) Prove the set $\text{Aut}(G)$ of all automorphisms of a group $G$ forms a group, the binary operation being the composition of functions.
   
   (b) Determine the group of automorphisms of each of the following groups.
      
      i. $(\mathbb{Z}, +)$ (also known as $\mathbb{Z}^+$)
      
      ii. A cyclic group of order 10.
      
      iii. $S_3$

2. Do one of the following.
   
   (a) Describe all homomorphisms $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$. Determine which are one-to-one, which are onto and which are isomorphisms.

   (b) Do all of the following.
      
      i. Prove that if a group contains exactly one element of order 2, then that element is in the center of the group.
      
      ii. Suppose $\phi : G \to G'$ is an onto homomorphism. Prove, if $G$ is cyclic, then $G'$ is cyclic.
      
      iii. Suppose $\phi : G \to G'$ is an onto homomorphism. Prove, if $G$ is abelian, then $G'$ is abelian.

3. Do either of the following.
   
   (a) Find all subgroups of $S_3$ and determine which of these are normal.

   (b) Find all subgroups of the quaternion group and determine which of these are normal.

4. Let $\phi : G \to G'$ be an onto homomorphism and let $N$ be a normal subgroup of $G$.
   
   (a) i. Show that the set $\phi(N) = \{\phi(n) : n \in N\}$ is a subgroup of $G'$.
      
      ii. Prove that $\phi(N)$ is a normal subgroup of $G'$. 