"Iron rusts from disuse; stagnant water loses its purity and in cold weather becomes frozen; even so does inaction sap the vigor of the mind.” – Leonardo da Vinci

Problems

1. Do one of the following.

   (a) Let $G = D_4$ be the dihedral group of symmetries of the square.
       i. What is the stabilizer of a vertex? Of an edge?
       ii. $G$ acts on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?

   (b) Let $G = GL(n, R)$ operate on the set $S = R^n$ by left multiplication.
       i. Describe the decomposition of $S$ into orbits for this operation.
       ii. What is the stabilizer of $e_1$?

2. Do one of the following.

   (a) Let $G$ be a group and let $H$ be the cyclic subgroup generated by an element $x$ of $G$. Show that if left multiplication by $x$ fixes every coset of $H$ in $G$, then $H$ is a normal subgroup of $G$.

   (b) A map $\phi : S \to S'$ of $G$-sets is called a homomorphism of $G$-sets if $\phi(gs) = g\phi(s)$ for all $s \in S$ and all $g \in G$. Let $\phi$ be such a homomorphism. Prove the following.
       i. The stabilizer $G_{\phi(s)}$ contains the stabilizer $G_s$.
       ii. The orbit of an element $s \in S$ maps onto the orbit of $\phi(s)$.

3. Let $G$ be the group of rotational symmetries of a cube $C$. Two regular tetrahedra $\Delta$ and $\Delta'$ can be inscribed in $C$, each using half of the vertices. What is the order of the stabilizer of $\Delta$?