Do any two (2) of these computational problems

C.1. Show that \[
\begin{bmatrix}
-1 \\
1 \\
2
\end{bmatrix}
\] is an eigenvector for the matrix \[
\begin{bmatrix}
2 & -6 & 6 \\
1 & 9 & -6 \\
-2 & 16 & -13
\end{bmatrix}
\] and determine the corresponding eigenvalue.

(a) \[
\begin{bmatrix}
2 & -6 & 6 \\
1 & 9 & -6 \\
-2 & 16 & -13
\end{bmatrix}
\begin{bmatrix}
-1 \\
1 \\
2
\end{bmatrix} = \begin{bmatrix}
4 \\
-4 \\
-8
\end{bmatrix} = -4 \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}
\] so the eigenvalue is \( \lambda = -4 \).

C.2. Given the subspace \( V \) of \( \mathbb{C}^4 \) where \( V = \langle \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \rangle \), determine the dimension of the subspace \( V^\perp \) by finding a basis for \( V^\perp \).

(a) \[
V^\perp = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix} = 0 \right\}
\]

\[
= \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + b = 0 \text{ and } a + 2b + 3c + 4d = 0 \right\}
\]

\[
= \left\{ \begin{bmatrix} 3c + 4d \\ -3c - 4d \\ c \\ d \end{bmatrix} \right\} = \left\{ c \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \end{bmatrix} : c, d \in \mathbb{C} \right\}
\]

so \( \left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\} \) is a basis for \( V^\perp \) and the latter has dimension 2.
C.3. The characteristic polynomial of $A = \begin{bmatrix} -2 & -6 & -6 \\ -3 & 2 & -2 \\ 3 & 2 & 6 \end{bmatrix}$ is $P_A(x) = -(x + 2)(x - 4)^2$. Find all eigenvalues and determine their algebraic and geometric multiplicities.

(a) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $A - (-2)I_3 = \begin{bmatrix} 0 & -6 & -6 \\ -3 & 4 & -2 \\ 3 & 2 & 8 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $E_A(-2) = \langle \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \rangle$

$A - 4I_3 = \begin{bmatrix} -6 & -6 & -6 \\ -3 & -2 & -2 \\ 3 & 2 & 2 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $E_A(4) = \langle \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \rangle$

Thus, $\lambda = -2$ has algebraic multiplicity 1 and geometric multiplicity 1 and $\lambda = 4$ has algebraic multiplicity 2 and geometric multiplicity 1.

Do any two (2) of these problems from the text, homework, or class.

You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Prove Theorem RMRT, Rank of a Matrix is the Rank of the Transpose:

Suppose $A$ is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

(a) The proof is in the textbook.

M.2. From Project 11: Explain why the following $5 \times 5$ matrix that has a $3 \times 3$ zero submatrix is definitely singular (regardless of the 16 non-zeros marked by $x$’s.)

$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$

(a) We show $\det(A) = 0$ which implies $A$ is singular. Note that expanding $\det = \begin{bmatrix} x & x & x & 0 & 0 & x & x \\ 0 & 0 & x & x & 0 & 0 & x & x \\ 0 & 0 & x & x & 0 & 0 & x & x \end{bmatrix}$ along the first column gives $x \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$ which equals zero because of the column of all zeros.

Thus, expanding the determinant of $A$ along the top row gives

$$\det(A) = x \begin{vmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{vmatrix} - x \begin{vmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{vmatrix} + 0 - 0$$

$$= 0 - 0$$

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M.3. Exercise T60 in subsection PD (Properties of Dimension): Suppose that $W$ is a vector space with dimension 5, and $U$ and $V$ are subspaces of $W$, each of dimension 3. Prove that $U \cap V$ contains a non-zero vector.

(a) Proof in text.

Do two (2) of these problems you’ve not seen before.

T.1. Label the following statements as being true or false.

(a) The rank of a matrix is equal to the number of its nonzero columns. \[\text{False: } \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\]

(b) The rank of a matrix is equal to the number of its nonzero rows. \[\text{False: } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\]

(c) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0. \[\text{True}\]

(d) Elementary row operations preserve rank. \[\text{True}\]

(e) An $n \times n$ matrix of rank $n$ is invertible. \[\text{True}\]

(f) It is possible for a $3 \times 5$ matrix to have rank 4. \[\text{False: a } 3 \times 5 \text{ can have at most 3 leading ones}\]

(g) It is possible for a $5 \times 3$ matrix to have rank 4. \[\text{False: a } 5 \times 3 \text{ can have at most 3 leading ones}\]

T.2. Suppose that $A$ is a $4 \times 4$ matrix with exactly two distinct eigenvalues, 5 and $-9$ and let $E_A (5)$ and $E_A (-9)$ be the corresponding eigenspaces, respectively.

(a) Write all possible characteristic polynomials of $A$ that are consistent with $\dim (E_A (5)) = 3$.

i. $1 \leq \gamma_A (\lambda) \leq \alpha_A (\lambda)$ and $\alpha_A (5) + \alpha_A (-9) = 4$ tells us that $P_A (x) = (x-5)^3 (x+9)^1$

(b) Write all possible characteristic polynomials of $A$ that are consistent with $\dim (E_A (-9)) = 2$.

i. $1 \leq \gamma_A (\lambda) \leq \alpha_A (\lambda)$ and $\alpha_A (5) + \alpha_A (-9) = 4$ tells us that $P_A (x) = (x-5)^2 (x+9)^2$ or $P_A (x) = (x-5)^1 (x+9)^3$

T.3. A matrix $A$ is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

(a) $A^2 \vec{x} = A (A \vec{x}) = A (\lambda \vec{x}) = \lambda (A \vec{x}) = \lambda (\lambda \vec{x}) = \lambda^2 \vec{x}$ and $A^2 \vec{x} = A \vec{x} = \lambda \vec{x}$ tells us $\lambda^2 \vec{x} = \lambda \vec{x}$ and since $\vec{x} \neq 0$, then $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is idempotent and with eigenvalues 0 and 1.

T.4. An $n \times n$ matrix $A$ is \textbf{nilpotent} if, for some positive integer $k$, $A^k = O$, where $O$ denotes the $n \times n$ zero matrix. Prove that if $A$ is nilpotent, then $A$ is not invertible.

(a) Consider $0 = \det (O) = \det (A^k) = [\det (A)]^k$. Thus $\det (A) = 0$ and $A$ is singular and hence not invertible.