C.1. Do all of the following.

(a) Show that the set of vectors \( S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\} \) is linearly dependent.

i. \( A = \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 2 & 1 \end{bmatrix} \) has row echelon form: \( B = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \), Thus the homogenous linear system
\[
\begin{align*}
x_1 &+ x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]
and so there are nontrivial solutions. Any such solution (like \( x_1 = 5, x_2 = 0, x_3 = -3, x_4 = 1 \)) gives a nontrivial relation of linear dependence for the vectors in \( S \) making \( S \) linearly dependent.

(b) Find two vectors \( \vec{w}_1, \vec{w}_2 \) that are both in \( S \) and for which \( < S > = < T > \), where \( T = \{ \vec{w}_1, \vec{w}_2 \} \).

i. We can’t find two vectors whose span equals the span of \( S \) but we can find three. By throwing out the last vector in \( S \) (because it is associated with a free variable), we get
\( T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\} \) and Theorem BS tells us that \( < S >= < T > \).

(c) Write one of the extra vectors in \( S \) as a linear combination of \( \vec{w}_1, \vec{w}_2 \).

i. Using our solutions from part (a) we see
\[
\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]

C.2. Write all of the following complex numbers in the form \( a + bi \).

(a) \( 2(2 - 3i) - 7(6 + 2i) = -38 - 20i \)

(b) \( \frac{4 + 3i}{2 - i} = \frac{4 + 3i}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{10 + 5i}{5} = 2 + i \)

(c) \( \sqrt[5]{i} \) [Hint: write \( (a + bi)^2 = i \) and solve a system of equations.]

i. \( (a + bi)^2 = i \) gives \( a^2 - b^2 + 2abi = 0 + i \)

ii. so \( a^2 - b^2 = 0, \) and \( 2ab = 1. \)
iii. $a = \pm b$ and substituting gives $\pm 2b^2 = 1$. Using the plus sign we have $b = \frac{1}{\sqrt{2}}$ and choosing $a = b = \frac{1}{\sqrt{2}}$ we see that one square root of $i$ is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$

C.3. The vectors $\vec{u}_1, \vec{u}_2$, and $\vec{u}_3$ below are already orthonormal. Use the Gram-Schmidt procedure to find a vector $\vec{u}_4$ so that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthonormal set.

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find all vectors $\vec{v}_4$ in $\mathbb{R}^4$ so that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ form an orthonormal set.

(a) The Gram-Schmidt formula is

$$\vec{u}_i = \vec{v}_i - \left( \frac{\langle \vec{v}_i, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \right) \vec{u}_1 - \cdots - \left( \frac{\langle \vec{v}_i, \vec{u}_{i-1} \rangle}{\langle \vec{u}_{i-1}, \vec{u}_{i-1} \rangle} \right) \vec{u}_{i-1}$$

so

$$\vec{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left( \frac{\langle \vec{v}_4, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \left( \frac{\langle \vec{v}_4, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} - \left( \frac{\langle \vec{v}_4, \vec{u}_3 \rangle}{\langle \vec{u}_3, \vec{u}_3 \rangle} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left( \frac{1}{2} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \left( \frac{1}{2} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} - \left( \frac{1}{2} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ -1/4 \\ -1/4 \end{bmatrix}$$

(b) To guarantee the vectors are orthonormal, we divide by $\langle \vec{u}_4, \vec{u}_4 \rangle = \sqrt{4/16} = \frac{1}{2}$ giving a new

$$\vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}.$$ 

C.4. Compute the following matrix-vector product by hand in two ways.

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}.$$ 

(a) Using term by term multiplication:

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -10 & 2 \\ 10 & 5 \end{bmatrix}.$$ 

(b) Using the definition:

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 2 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}.$$
Do any two (2) of these problems from the text, homework, or class.

You may NOT just cite a theorem or result in the text. You must prove these results.

M.1. Suppose \( S = \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p \} \) is a linearly independent set and that \( \vec{v} \notin < S > \). Prove the set \( W = \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p, \vec{v} \} \) is a linearly independent set.

(a) Using the definition. Let
\[
\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_p \vec{u}_p + \alpha_{p+1} \vec{v} = \vec{0} \tag{1}
\]
be a relation of linear dependence. We show that the only way this equation can be true is if all of the \( \alpha \)'s equal 0.

(b) If \( \alpha_{p+1} \neq 0 \) then we can write \( \vec{v} \) as a linear combination of the other vectors \( \vec{v} = \frac{-\alpha_1}{\alpha_{p+1}} \vec{u}_1 - \cdots - \frac{-\alpha_p}{\alpha_{p+1}} \vec{u}_p. \) But we know \( \vec{v} \) is not in the span of \( S \) so this is impossible. Hence we can conclude that \( \alpha_{p+1} \) must be zero in equation (1.) and so that equation can be rewritten as
\[
\vec{0} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_p \vec{u}_p + (0) \vec{v} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_p \vec{u}_p
\]
(c) Now the linear independence of \( S \) tells us the rest of the \( \alpha \)'s are also 0 and we are done.

M.2. Suppose \( S = \{ \overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3 \} \) is a linearly independent set in \( \mathbb{R}^3 \). Is the set of vectors \( 2 \overrightarrow{v}_1 + \overrightarrow{v}_2 + 3 \overrightarrow{v}_3, \overrightarrow{v}_2 + 5 \overrightarrow{v}_3, 3 \overrightarrow{v}_1 + \overrightarrow{v}_2 + 2 \overrightarrow{v}_3 \) linearly dependent or independent?

(a) Using the definition we consider an arbitrary relation of linear dependence and then re-write it by collecting on \( \overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3 \)
\[
\overrightarrow{0} = a (2 \overrightarrow{v}_1 + \overrightarrow{v}_2 + 3 \overrightarrow{v}_3) + b (\overrightarrow{v}_2 + 5 \overrightarrow{v}_3) + c (3 \overrightarrow{v}_1 + \overrightarrow{v}_2 + 2 \overrightarrow{v}_3)
\]
\[
= (2a + 0b + 3c) \overrightarrow{v}_1 + (a + b + c) \overrightarrow{v}_2 + (3a + 5b + 2c) \overrightarrow{v}_3
\]
(b) Since \( S \) is linearly independent we know each of the coefficients above must equal zero giving rise to the homogeneous system of linear equations with augmented matrix \( [A] \overrightarrow{0} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 5 & 2 & 0 \end{bmatrix} \). Row reducing we obtain \( B = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Thus we see that the linear system has infinitely many solutions – one of which is \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \). This means \( S \) is linearly dependent since there are many non-trivial relations of linear dependence, one of which is
\[
\overrightarrow{0} = -3 (2 \overrightarrow{v}_1 + \overrightarrow{v}_2 + 3 \overrightarrow{v}_3) + (1) (\overrightarrow{v}_2 + 5 \overrightarrow{v}_3) + 2 (3 \overrightarrow{v}_1 + \overrightarrow{v}_2 + 2 \overrightarrow{v}_3)
\]


Suppose that \( A \) and \( B \) are \( m \times n \) matrices. Then \( (A + B)^t = A^t + B^t \).

(a) This proof is in the textbook on page 206.
Do one (1) of these problems you’ve not seen before.

T.1. Suppose $A$ is a square matrix of size $n$ satisfying $A^2 = AA = O$. Prove that the only vector $\vec{x}$ satisfying $(I_n - A)\vec{x} = \vec{0}$ is the zero vector.

(a) We algebraically manipulate the equation

\[
(I_n - A)\vec{x} = \vec{0} \\
A (I_n - A)\vec{x} = A\vec{0} \\
(\begin{bmatrix} I_n - A \end{bmatrix}) \vec{x} = \vec{0} \\
A\vec{x} - A^2\vec{x} = \vec{0} \\
A\vec{x} - O\vec{x} = \vec{0} \\
A\vec{x} - \vec{0} = \vec{0} \\
A\vec{x} = \vec{0}
\]

T.2. Recall that

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Now explain why the fact that

\[
\begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -4 & -2 & -2 & 0 & 1 & 0 \\ -5 & -2 & -4 & 0 & 0 & 1 \end{bmatrix}
\]

has reduced row-echelon form

\[
\begin{bmatrix} 1 & 0 & 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 0 & -\frac{5}{2} & 2 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}
\]

tells us the only vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ that can be in the span of $S = \left\{ \begin{bmatrix} 3 \\ -4 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix} \right\}$ are those where $a + 2b - c = 0$.

(a) We know that $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the span of $S$ if and only if the following system of linear equations is consistent

\[
\begin{align*}
3x + 2y + 0z &= a \\
-4x - 2y - 2z &= b \\
-5x - 2y - 4z &= c
\end{align*}
\]

(b) But we can write this as

\[
\begin{align*}
3x + 2y + 0z &= (1) a + (0) b + (0) c \\
-4x - 2y - 2z &= (0) a + (1) b + (0) c \\
-5x - 2y - 4z &= (0) a + (0) b + (1) c
\end{align*}
\]

(c) Now note that running elementary operations on the matrix $B$ below uses the last three columns to

\[
B = \begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -4 & -2 & -2 & 0 & 1 & 0 \\ -5 & -2 & -4 & 0 & 0 & 1 \end{bmatrix}
\]

keep track of how many $a$, $b$, and $c$’s there are if we were to run those same elementary row operations by hand on the augmented matrix $A$.

\[
A = \begin{bmatrix} 3 & 2 & 0 & a \\ -4 & -2 & -2 & b \\ -5 & -2 & -4 & c \end{bmatrix}
\]

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Thus the reduced echelon form of $B$ tells us that the reduced row echelon form of $A$ is

\[
\begin{bmatrix}
1 & 0 & 2 & (0) a + b - c \\
0 & 1 & -3 & (0) a - \frac{5}{2} b + 2c \\
0 & 0 & 0 & a + 2b - c
\end{bmatrix}
\]

which represents a consistent system if and only if $a + 2b - c$ is not 0.