Directions:

- Only write on one side of each page.
- Use terminology correctly.
- Partial credit is awarded for correct approaches so justify your steps.

Do both of these ”Computational” problems

C.1. [15 points] If $V$ is a subspace of $\mathbb{C}^n$ then $V^\perp$ is defined to be the set $V^\perp = \{ \vec{x} \in \mathbb{C}^n \mid \forall \vec{v} \in V \langle \vec{x}, \vec{v} \rangle = 0 \}$. That is, $V^\perp$ is the set of all vectors in $\mathbb{C}^n$ that are orthogonal to every vector in $V$.

1. Show that $V^\perp$ is a subspace of $\mathbb{C}^n$.

C.2. [15 points] Express $4 - t - t^2$ as a linear combination of the vectors in $S = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$.

Do one (1) of these ”In Class, Text, or Homework” problems

1. [15 points] Show that $C(AB) \subseteq C(A)$. Here, $C(A)$ is the column space of matrix $A$.

2. [15 points] Prove that if matrix $A$ is diagonalizable then $A^3$ is diagonalizable.

Do any two (2) of these ”Other” problems

1. [20 Points] Prove that if $A, B$ are matrices for which the product $AB$ is defined, then $\eta(B) \leq \eta(AB)$. Here $\eta(A)$ is the nullity of $A$.

2. [20 Points] Let $A$ be an $n \times n$ matrix and let $\lambda$ be a nonzero eigenvalue of $A$. Show that if $\vec{x}$ is an eigenvector corresponding to $\lambda$ then $\vec{x}$ is in the column space of $A$.

3. [20 Points] Prove the following by contradiction. If $\lambda$ and $\rho$ are two distinct (not equal) eigenvalues of the square matrix $A$, then the intersection of the eigenspaces for these two eigenvalues is trivial. That is, $E_A(\lambda) \cap E_A(\rho) = \{0\}$.

Definitions

1. [15 points] Given a set $V$ and an addition and scalar multiplication for elements in $V$, there are 10 properties that must hold for $V$ to be a vector space. List those properties. Give the actual mathematical statements of the properties rather than the names of the properties. For example: write $\alpha (\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ instead of saying “scalar multiplication distributes over vector addition”.

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Useful information

1. $\bar{x} \in N(A)$ iff $A\bar{x} = \bar{0}$

2. $\bar{y} \in C(A)$ iff there exists an $\bar{x}$ with $A\bar{x} = \bar{y}$