D.1. [1 point each] Given a set $V$ and an addition and scalar multiplication for elements in $V$, there are 10 properties that must hold for $V$ to be a vector space. List those properties. Give mathematical statements of the properties not the names of the properties.

Do two (2) of these ”Computational” problems

C.1. [15 points] Show that the set of all vectors in $\mathbb{C}^4$ whose coordinates sum to zero is a subspace of $\mathbb{C}^4$.

C.2. [15 points] Prove that the set $S = \{1 + t^2, t + t^2, 1 + 2t - t^2\}$ is a basis for $P_2$.

C.3. [15 points] Let $V$ be the subspace of $\mathbb{C}^4$ with basis $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Find a basis for $V^\perp$.

Do any two (2) of these ”In Class, Text, or Homework” problems

M.1. [8, 7 points each] Do both of the following.

1. Give an example of a $2 \times 2$ matrix $A$ for which $A \neq O_2$ and $A^2 = O_2$.
2. Given that $A$ and $B$ are both $n \times n$ matrices. Prove that if $\vec{x}$ is an eigenvector for both $A$ and $B$, then $\vec{x}$ is an eigenvector for $A + B$.

M.2. [15 points] Given a basis $\{\vec{v}_1, \cdots, \vec{v}_r\}$ of $\mathbb{C}^n$ and a matrix $A$ for which $\{\vec{v}_{r+1}, \cdots, \vec{v}_n\}$ is a basis for the null space of $A$. Prove the set $\{A\vec{v}_1, \cdots, A\vec{v}_r\}$ spans the column space of $A$.

M.3. [15 points] Let $U$ be a vector space and $V, W$ subspaces of $U$ of dimension 2 and 3 respectively. Let $B = \{\vec{v}_1, \vec{v}_2\}$ be a basis of $V$ and $C = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ a basis for $W$. If the only vector common to both $V$ and $W$ is $\vec{0}$ prove that the set $B \cup C = \{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly independent.
Do any two (2) of these "Other" problems

T.1. [12, 3 points each] A square matrix of size $n$ is said to be nilpotent if there is a positive integer $s$ for which $A^s = O_n$. Prove that if $A$ is nilpotent, then

1. $\lambda = 0$ is an eigenvalue of $A$ and is the only eigenvalue of $A$.
2. Use this information to determine the characteristic polynomial of $A$, $p_A(x)$.

T.2. [15 points] Prove that if $U$ and $W$ are both subspaces of a vector space $V$ then the intersection, $U \cap W$, is also a subspace of $V$.

T.3. [3 points each] Suppose $V$ is a vector space of dimension 7 and $W$ is a subspace of dimension 4.

1. (a) True or False
   (b) Every basis for $W$ can be extended to a basis for $V$ by adding three more vectors.
   (c) Every basis for $V$ can be reduced to a basis for $W$ by removing three vectors.
   
Suppose now that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_7, \vec{v}_8, \vec{v}_9$ are nine vectors in $\mathbb{C}^7$.
   
Choose your answer.
   (d) Those vectors (are)(are not)(might be) linearly independent.
   (e) They (do)(do not)(might) span $\mathbb{C}^7$.
   (f) If those vectors are the columns of matrix $A$, then $A\vec{x} = \vec{b}$ (has)(does not have)(might not have) a solution.

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